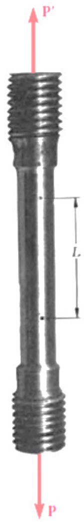
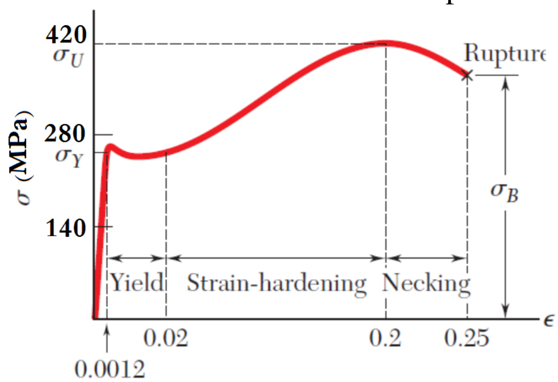
CHAPTER 2

Failure/Fracture Criterion

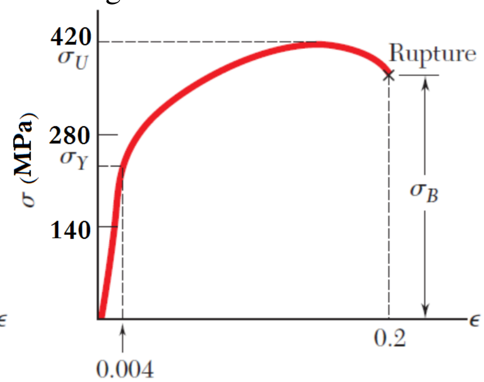
Failure (Yield) Criteria for Ductile Materials under Plane Stress



Designer engineer: 1- Analysis of loading (for simple geometry using what you learn here and for complex ones using finite element approach) and **2-** Does failure occur under the predicted loading condition?

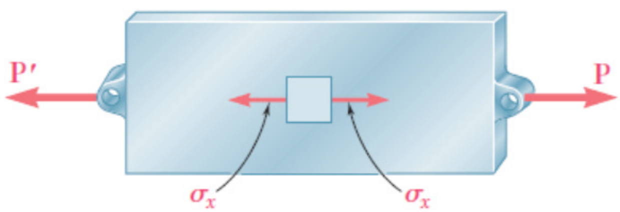


(*a*) Low-carbon steel

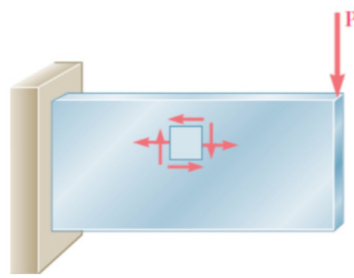


(*b*) Aluminum alloy

For uniaxial stress: $\Theta p = 0$ & $\Theta s = \pm 45°$: shear is responsible for the failure of ductile materials ($\tau_{max)Y} = \sigma_Y/2$)



Uniaxial Stress ($\sigma_x < \sigma_Y$)



Plane Stress (?)

Different criterions for ductile (Tresca & von Mises) and brittle **conditions** (Coulomb & Mohr)

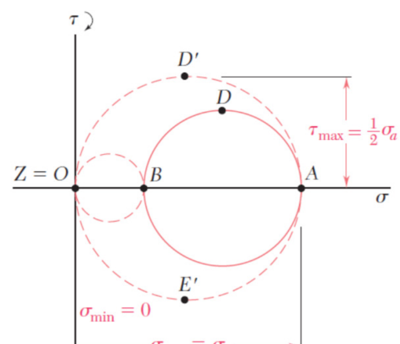
Maximum-Shearing-Stress Criterion for Ductile Materials (Tresca, 1868)

$\tau_{max} < \tau_{max)Y}$ (at uniaxial test) $\tau_{max)Y} = \sigma_Y/2$ therefore $\tau_{max} < \sigma_Y/2$

It is more accurate to use experimental data for τ_y

$\tau_{max} = \sigma_a/2$ **if principal stresses have the same signs**

$\tau_{max} = |\sigma_a - \sigma_b|/2$ **if principal stresses have opposite signs**



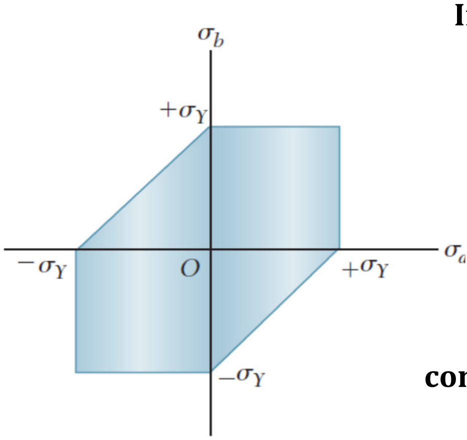
If the principal stresses σa and σb have the same sign

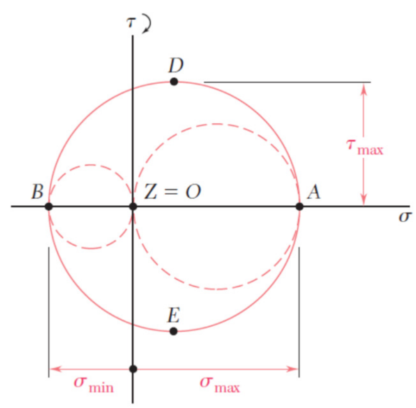
$$|\sigma_a| < \sigma_y \quad \text{and} \quad |\sigma_b| < \sigma_y$$

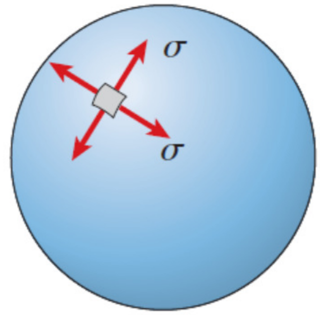
If they have opposite signs

$$|\sigma_a - \sigma_b| < \sigma_y$$

Assumptions: only shear is considered (not sensitive to σ_a & σ_b), $\sigma_{yT} = \sigma_{yC}$

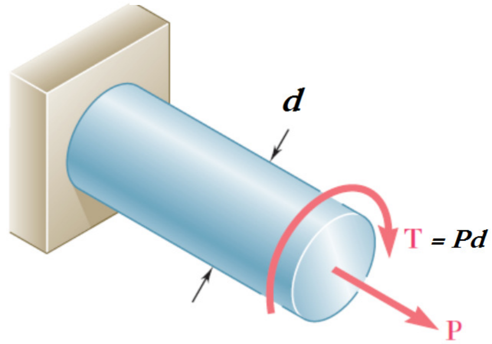






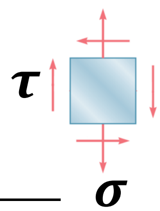
Example 4: Consider a spherical vessel with thickness of *t*, diameter of D, and yield stress of σ_y. At what value of internal pressure (*P*) will yielding occur according to Tresca's criterion?

$$\sigma_1 = \sigma_2 = \frac{Pr}{2t} \qquad \tau_{max} = \frac{\frac{Pr}{2t} - 0}{2} = \frac{Pr}{4t} = \frac{\sigma_y}{2} \qquad P = \frac{2t\sigma_y}{r} = \frac{4t\sigma_y}{D}$$



Example 5: Calculate allowable P according to Tresca's criterion and considering a factor of safety of 2.

$$\sigma = \frac{P}{\frac{\pi}{4}d^2} = \frac{4P}{\pi d^2} \qquad \tau = \frac{Tr}{J} = \frac{Pd\,\frac{d}{2}}{\frac{\pi}{2}\left(\frac{d}{2}\right)^4} = \frac{16P}{\pi d^2}$$



$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + {\tau_{xy}}^2} \qquad \sigma_{1,2} = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

$$\sigma_3 = 0$$

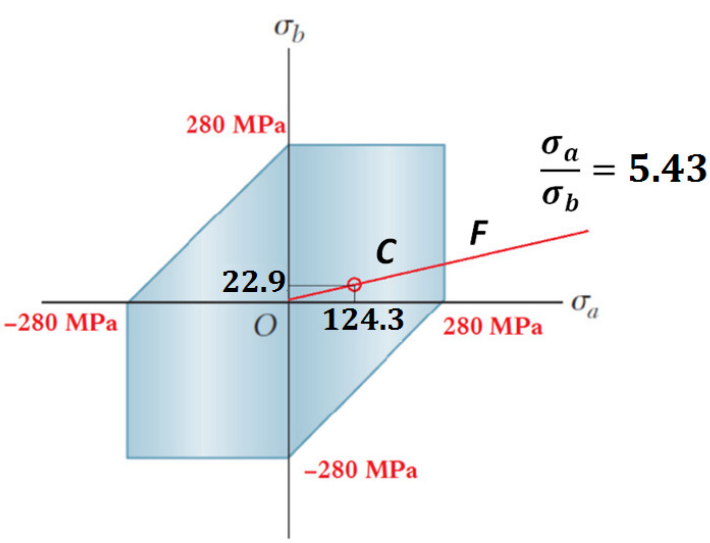
$$\sigma_{1,2} = \frac{2P}{\pi d^2} \pm \sqrt{\left(\frac{2P}{\pi d^2}\right)^2 + \left(\frac{16P}{\pi d^2}\right)^2} \qquad \sigma_1 = \frac{2P}{\pi d^2} + \frac{\sqrt{260}P}{\pi d^2} > 0 \quad \sigma_2 = \frac{2P}{\pi d^2} - \frac{\sqrt{260}P}{\pi d^2} < 0$$

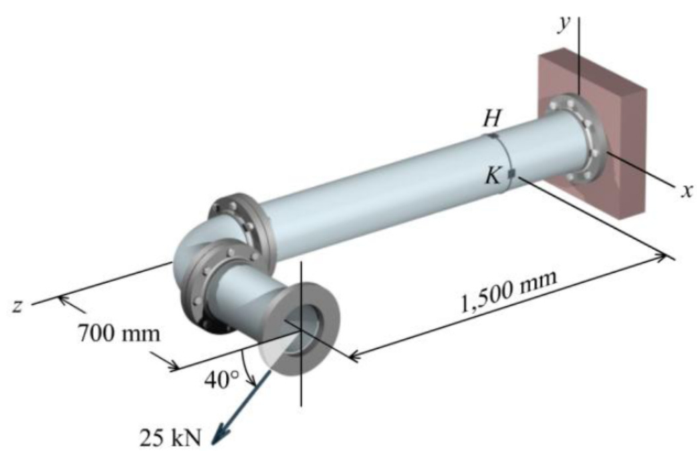
$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sqrt{260}P}{\pi d^2} \qquad SF = \frac{\tau_y}{\tau_{max}} = \frac{\frac{\sigma_y}{2}}{\frac{\sqrt{260}P}{\pi d^2}} = 2 \quad \rightarrow \quad P = \frac{\pi d^2 \sigma_y}{4\sqrt{260}}$$

Example 6: Calculate Factor of Safety according to the Tresca's criterion (σ_Y = 280 MPa)

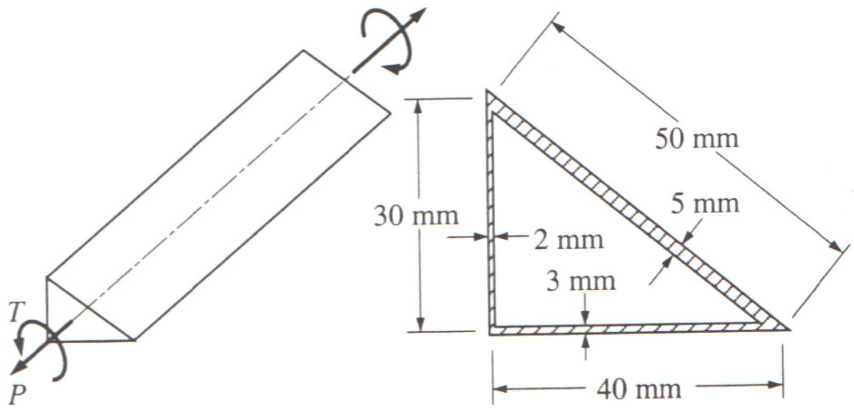
At point K: σ_{max} = 124.3 MPa σ_{min} = 22.9 MPa, $\tau_{max)abs} = \frac{124.3-0}{2} = 62.2\ MPa$

$$FS = \frac{\tau_Y}{\tau_{max}} = \frac{280/2}{62.2} = 2.25, \qquad FS = \frac{OF}{OC} = \frac{284.7}{126.4} = 2.25$$





TBR 4: P = 25 kN, σ_y = 420 MPa, Factor of Safety = 4. Find maximal T based on Tresca's criterion.



(a) (b)

$$\tau = \frac{T}{2\bar{A}t} \;\; and \;\; \sigma = \frac{P}{A}$$

$$\bar{A} = \frac{1}{2}\,(30\,mm)(40\,mm) = 600\,mm^2$$

$$\tau_{max} = \frac{T}{2\bar{A}t} = \frac{T}{2\,(600\,mm^2)(2\,mm)} = \frac{T}{2400}$$

$$\sigma = \frac{P}{A} = \frac{25000\,N}{(30\times 2 + 50\times 5 + 40\times 3)} = 58.14\,MPa$$

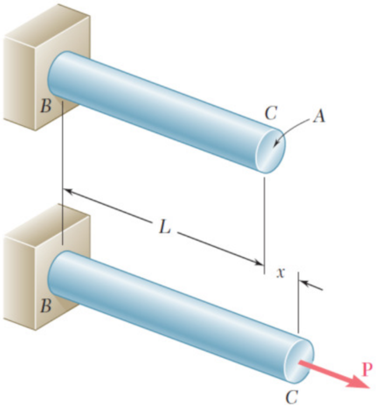
$$\sigma_{1,2} = \frac{\sigma}{2} \pm \sqrt{(\frac{\sigma}{2})^2 + \tau^2} = \frac{58.14}{2} \pm \sqrt{(\frac{58.14}{2})^2 + (\frac{T}{2400})^2}$$

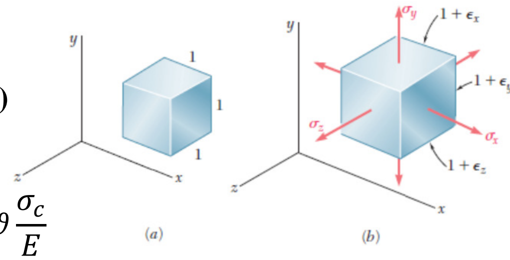
$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \sqrt{(\frac{58.14}{2})^2 + (\frac{T}{2400})^2}$$

$$SF = \frac{\tau_y}{\tau_{max}} = \frac{\frac{\sigma_y}{2}}{\sqrt{(\frac{58.14}{2})^2 + (\frac{T}{2400})^2}} = 4 \;\rightarrow T = 104923\,Nmm$$

$$T = 104.92\,Nm$$

Maximum-Distortion-Energy Criterion (von Mises' criterion)





Reminding points:

1) *Hook's Law:* $\varepsilon_a = \frac{\sigma_a}{E} - \vartheta\frac{\sigma_b}{E} - \vartheta\frac{\sigma_c}{E}$

2) *change in volume of a cube unit element:* $e = \varepsilon_x + \varepsilon_y + \varepsilon_z$

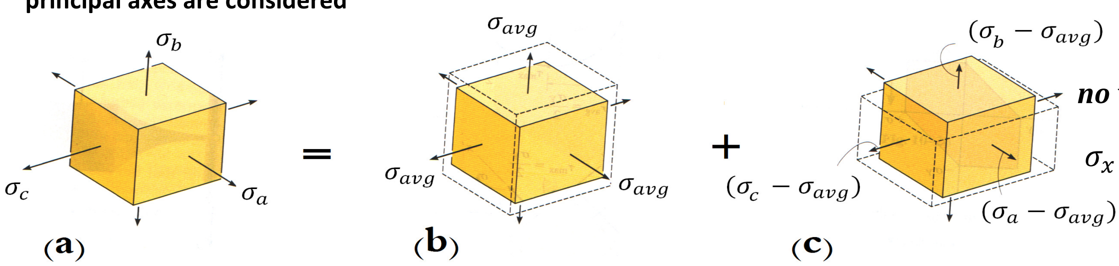
If: e = 0 then $\varepsilon_x + \varepsilon_y + \varepsilon_z = 0 \quad \rightarrow \quad \sigma_x + \sigma_y + \sigma_z = 0$

3) $U = \frac{1}{2}kx^2 = \frac{1}{2}Px \qquad u = \frac{U}{V} = \frac{1}{2}\sigma\varepsilon$ = ***strain-energy density***

$u = \frac{1}{2}\sigma_a\varepsilon_a + \frac{1}{2}\sigma_b\varepsilon_b + \frac{1}{2}\sigma_c\varepsilon_c$ Replace strains using Hook's law

Energy is a scalar quantity and thus for the sake of ease principal axes are considered

$$u = \frac{1}{2E}[\sigma_a^2 + \sigma_b^2 + \sigma_c^2 - 2\vartheta(\sigma_a\sigma_b + \sigma_a\sigma_c + \sigma_c\sigma_b)]$$



no volume change

$\sigma_x + \sigma_y + \sigma_z = 0$

$u = u_v + u_d \qquad \sigma_{avg} = \frac{\sigma_a + \sigma_b + \sigma_c}{3} \qquad \sigma_{\acute{a}} = \sigma_a - \sigma_{avg} \quad \sigma_{\acute{b}} = \sigma_b - \sigma_{avg} \quad \sigma_{\acute{c}} = \sigma_c - \sigma_{avg}$

Experimental evidence show that materials can withstand very high uniform (hydrostatic) stresses without yielding. In 1904 Huber proposed that yielding in a ductile material occurs when distortion energy per unit volume of the material exceeds or equals the distortion energy per unit volume of the same material when it is subjected to yielding in a sample tension test. To obtain u_d then substitute stresses with $\sigma_{a,b,c} - \sigma_{avg}$

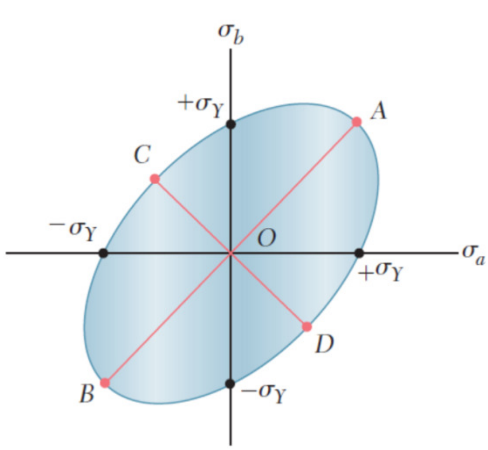
$u_d = \frac{1+\vartheta}{6E}\left[(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2\right] \qquad$ *for plane stress:* $\quad u_d = \frac{1+\vartheta}{3E}\left[\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2\right]$

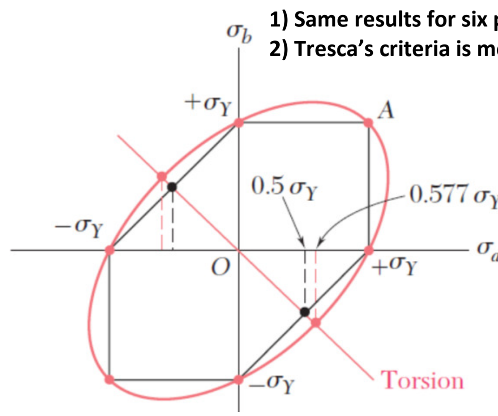
For uniaxial tension test $\sigma_a = \sigma_Y$ *and* $\sigma_b = 0 \quad (u_d)y = \frac{1+\vartheta}{3E}\sigma_Y^2 \quad$ *for plane stress:* $\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 = \sigma_Y^2$

For general state of stress: von Mises equivalent stress $= \sigma_M = \sqrt{\frac{1}{2}\left[(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2\right]} = \sigma_y$

Since only differences of the stresses are involved adding a constant stress to each does not alter the yield condition

1) Same results for six points (uniaxial and $\sigma_1 = \sigma_2$ conditions)
2) Tresca's criteria is more conservative





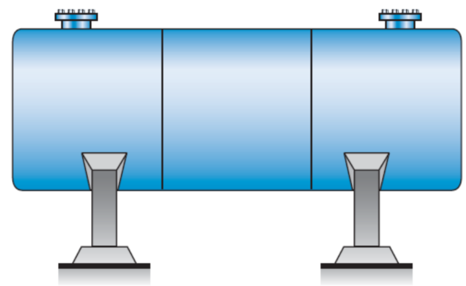
For torsion (maximum difference):

$\sigma_a = -\sigma_b = \pm 0.5\sigma_Y$ (Tresca)

$\sigma_a = -\sigma_b = \pm 0.577\sigma_Y$ (Mises)

$\tau_Y/\sigma_Y = 0.55$-0.6

Example 7: Consider a cylindrical vessel with internal pressure of *P*, radius of *r*, and yield stress of σ_y. Find minimal thickness of vessel based on von Mises criterion.

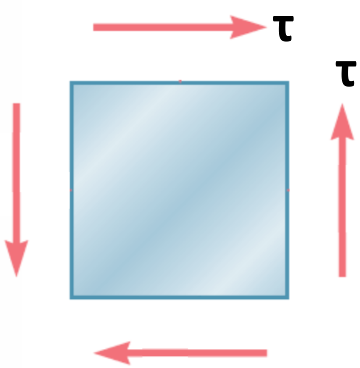


$$\sigma_1 = \frac{Pr}{t} \qquad \sigma_2 = \frac{Pr}{2t} \qquad \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2$$

$$\left(\frac{Pr}{t}\right)^2 - \left(\frac{pr}{t}\right)\left(\frac{Pr}{2t}\right) + \left(\frac{Pr}{2t}\right)^2 = \sigma_y^2 \qquad t = \frac{\sqrt{3}Pr}{2\sigma_y}$$

$$Tresca: \tau_{max} = \frac{\frac{Pr}{t} - 0}{2} = \frac{\sigma_y}{2} \to t = \frac{Pr}{\sigma_y}$$

Example 8: The element with yield stress of σ_y is under pure shear loading as shown. Find factor of safety based on Von-Mises criterion.

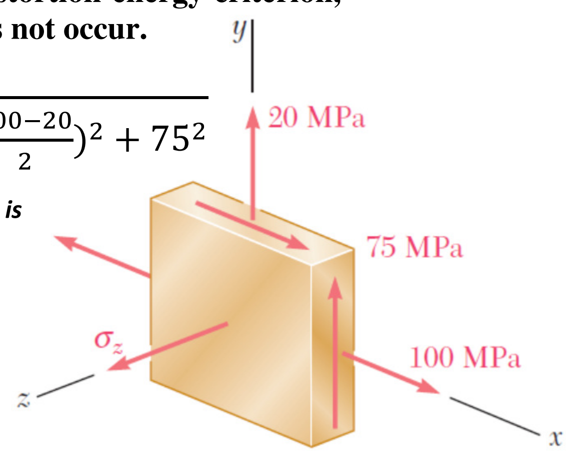


$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + {\tau_{xy}}^2} = \pm\,\tau$$

$$\sigma_v^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \qquad \sigma_v^2 = \tau^2 - \tau(-\tau) + \tau^2$$

$$\sigma_v = \sqrt{3}\,\tau \qquad SF = \frac{\sigma_y}{\sqrt{3}\,\tau} \; Based\ on\ Tresca\ criterion{:}\ SF = \frac{\tau_y}{\tau_{max}} = \frac{\frac{\sigma_y}{2}}{\tau} = \frac{\sigma_y}{2\tau}$$

Example 9: The state of stress shown occurs in a machine component made of a brass for which σ_Y = 160 MPa. Using the maximum-distortion-energy criterion, determine the range of values of σ_Z for which yield does not occur.



$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + {\tau_{xy}}^2} = \frac{100+20}{2} \pm \sqrt{\left(\frac{100-20}{2}\right)^2 + 75^2}$$

$\sigma_{1,2} = 145\ MPa, -25\ MPa$

σ_z* *is the 3rd principal stress as there is no shear stress on face z

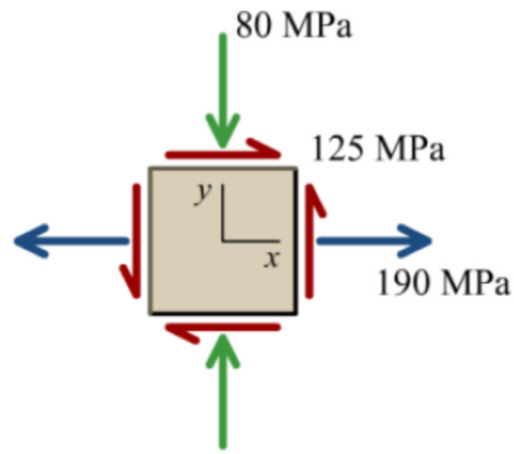
$$\sqrt{\frac{1}{2}\left[(\sigma_a-\sigma_b)^2 + (\sigma_b-\sigma_c)^2 + (\sigma_c-\sigma_a)^2\right]} = \sigma_y$$

$$\sqrt{\frac{1}{2}\left[(145-(-25))^2 + (-25-\sigma_z)^2 + (\sigma_z-145)^2\right]} < 160\ MPa$$

$$2\sigma_z^2 - 240\,\sigma_z - 650 < 0$$

$-2.65\ MPa < \sigma_z < 122.65\ MPa$ ***SOLVE USING TRESCA's CRITERIA. Is there a solution?***

TBR 5: The stresses on the surface of a hard bronze component are shown in the figure. The yield strength of the bronze is $\sigma_Y = 345$ MPa. (a) What is the factor of safety predicted by the maximum-shear-stress theory of failure for the stress state shown? Does the component fail according to this theory? (b) What is the value of the Mises equivalent stress for the given state of plane stress? (c) What is the factor of safety predicted by the failure criterion of the maximum distortion energy theory of failure? Does the component fail according to this theory?



Solution

Principal stresses:

$$\sigma_{p1,p2} = \frac{(190\text{ MPa}) + (-80\text{ MPa})}{2} \pm \sqrt{\left(\frac{(190\text{ MPa}) - (-80\text{ MPa})}{2}\right)^2 + (125\text{ MPa})^2}$$

$$= 55\text{ MPa} \pm 183.984\text{ MPa}$$

therefore,

$$\sigma_{p1} = 239\text{ MPa}$$

$$\sigma_{p2} = -129.0\text{ MPa}$$

(a) Maximum-Shear-Stress Theory: Since σ_{p1} is positive and σ_{p2} is negative, failure will occur if $|\sigma_{p1} - \sigma_{p2}| \geq \sigma_Y$. For the principal stresses existing in the component:

$$|\sigma_{p1} - \sigma_{p2}| = |238.984\text{ MPa} - (-128.984\text{ MPa})| = 367.968\text{ MPa} > 345\text{ MPa} \qquad \text{N.G.}$$

Therefore, **the component fails** according to the maximum-shear-stress theory. The factor of safety associated with this state of stress can be calculated as:

$$\text{FS} = \frac{345\text{ MPa}}{367.968\text{ MPa}} = \boxed{0.938} \qquad \textbf{Ans.}$$

(b) Mises equivalent stress: The Mises equivalent stress σ_M associated with the maximum-distortion-energy theory can be calculated from Eq. (15.8) for the plane stress state considered here.

$$\sigma_M = \left[\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2\right]^{1/2}$$

$$= \left[(238.984\text{ MPa})^2 - (238.984\text{ MPa})(-128.984\text{ MPa}) + (-128.984\text{ MPa})^2\right]^{1/2}$$

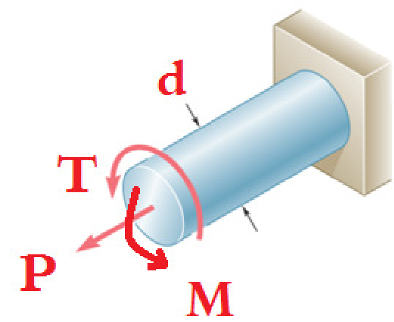
$$= 323.381\text{ MPa} = \boxed{323\text{ MPa}} \qquad \textbf{Ans.}$$

(c) Maximum-distortion-energy theory factor of safety: The factor of safety for the maximum-distortion-energy theory can be calculated from the Mises equivalent stress:

$$\text{FS} = \frac{345\text{ MPa}}{323.381\text{ MPa}} = \boxed{1.067} \qquad \textbf{Ans.}$$

According to the maximum-shear-stress theory, **the component does not fail.**

Example 10: The 60-mm-diameter shaft is made of a grade of steel with a 300 MPa tensile yield stress. Using the maximum-shearing-stress criterion and maximum-distortion-energy criterion determine the factor of safety for magnitude of the torque T= 5 kNm, *P* = 100 kN, and M = 2.3 kNm.



<u>Based on Tresca's criterion (for critical point):</u>

$$\sigma_x = \frac{P}{A} = \frac{100\,000\ N}{\frac{\pi}{4}(60\ mm)^2} = 35.36\ MPa, \quad \sigma_x = \frac{Mr}{I} = \frac{(2\,300\,000\ Nmm)\,(30\ mm)}{\frac{\pi}{4}(30\ mm)^4} = 108.46\ MPa$$

$$\tau_{xy} = \frac{Tr}{J} = \frac{(5\,000\,000\ Nmm)\,(30\ mm)}{\frac{\pi}{2}\ (30\ mm)^4} = 117.9\ MPa \qquad \sigma_x = 143.8\ MPa$$

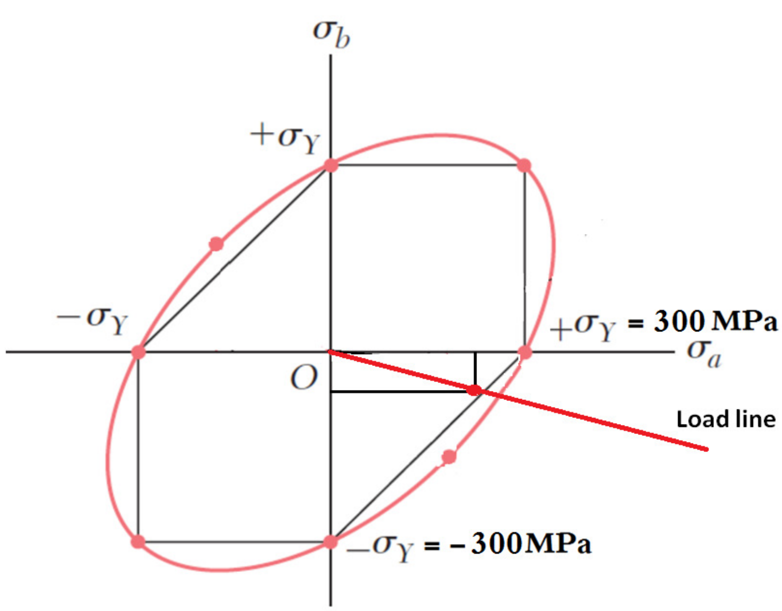
$$\sigma_{a,b} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{(\frac{\sigma_x-\sigma_y}{2})^2 + {\tau_{xy}}^2} \rightarrow \sigma_a = 210\ MPa\ ,\ \sigma_b = -66.2\ MPa$$

$$\tau_{max} = \frac{\sigma_a - \sigma_b}{2} = \frac{210-(-66.2)}{2} = 138.1 \qquad FS = \frac{\tau_y}{\tau_{max}} = \frac{\frac{300}{2}MPa}{138.1\ MPa} = 1.08$$

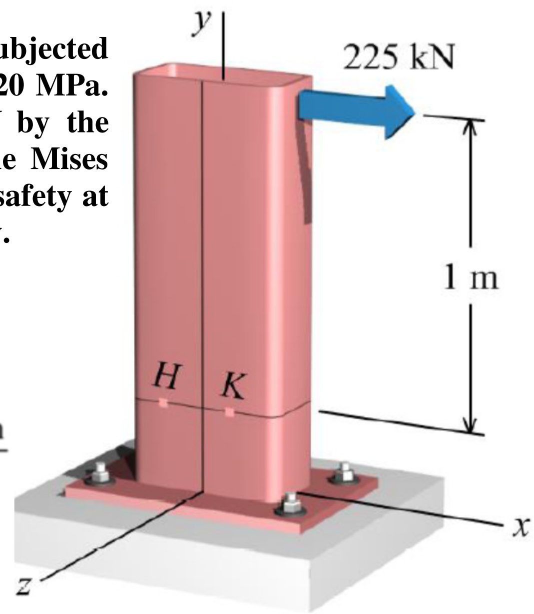
<u>Based on von Mises criterion:</u>

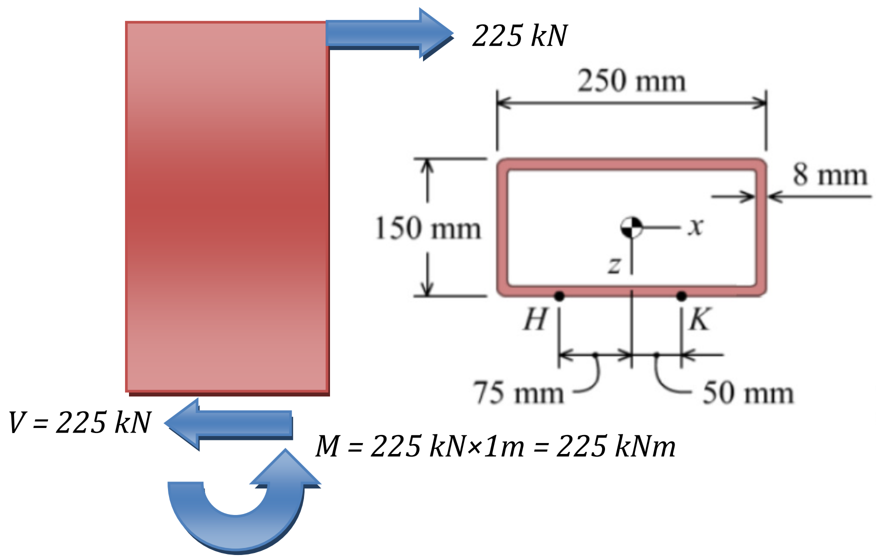
$$\sigma_M^2 = \sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 = (210)^2 - (-66.2\times 210) + (-66.2)^2$$

$$\sigma_M = 249.76\ MPa \qquad FS = \frac{300\ MPa}{249.76\ MPa} = 1.2 \qquad \frac{\sigma_a}{\sigma_b} = -3.2$$



Example 11: A hollow structural steel flexural member is subjected to the load shown. The yield strength of the steel is σ_Y = 320 MPa. (a) Determine the factors of safety predicted at point *K* by the maximum-shear-stress theory of failure. (b) Determine the Mises equivalent stresses at point *K*. (c) Determine the factors of safety at point *K* predicted by the maximum-distortion-energy theory.





$$\sigma_y = \frac{Mc}{I_z}, \qquad c = 50\,mm, \quad I_z = \frac{1}{12} \times 150 \times 250^3 - \frac{1}{12} \times (150-16)(250-16) = 5.22 \times 10^7\ mm^4$$

$$\rightarrow \sigma_y = \frac{(225 \times 10^6\ Nmm)(50\ mm)}{5.22 \times 10^7} = 215.37\ MPa\ \ (compressive) = -215.37\ MPa$$

$$\tau_{xy} = \frac{VQ_k}{I_z t_k}, \qquad t_k = 16\ mm,\ Q_k = \bar{y}A = (125-4)(8 \times 150) + 2\left[\frac{(125-50-8)}{2} + 50\right](67 \times 8)$$

$$= 234712\ mm^3$$

$$\rightarrow \tau_{xy} = \frac{VQ_k}{I_z t_k} = \frac{(225000\ N)(234712\ mm^3)}{(5.22 \times 10^7 mm^4)(16mm)} = 63.18\ MPa,\ \ \sigma_{a,b} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{(\frac{\sigma_x-\sigma_y}{2})^2 + {\tau_{xy}}^2}$$

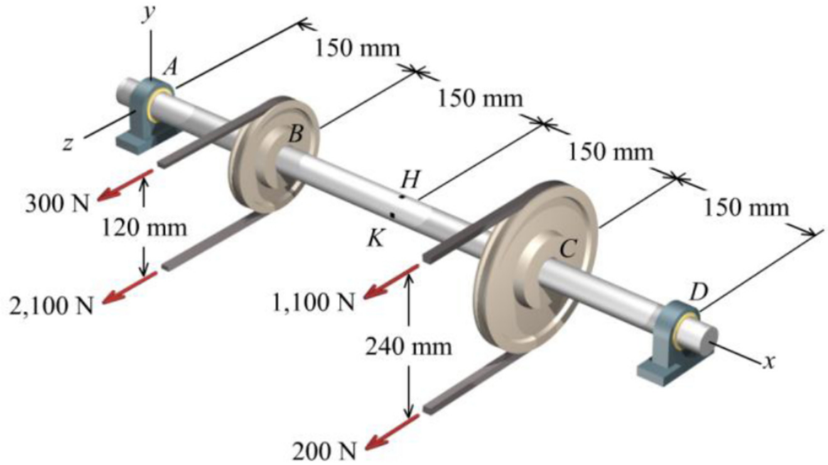
$$\sigma_{a,b} = \frac{0+(-215.37)}{2} \pm \sqrt{(\frac{0-(-215.37)}{2})^2 + 63.18^2} = 17.17\ MPa, -232.54\ MPa$$

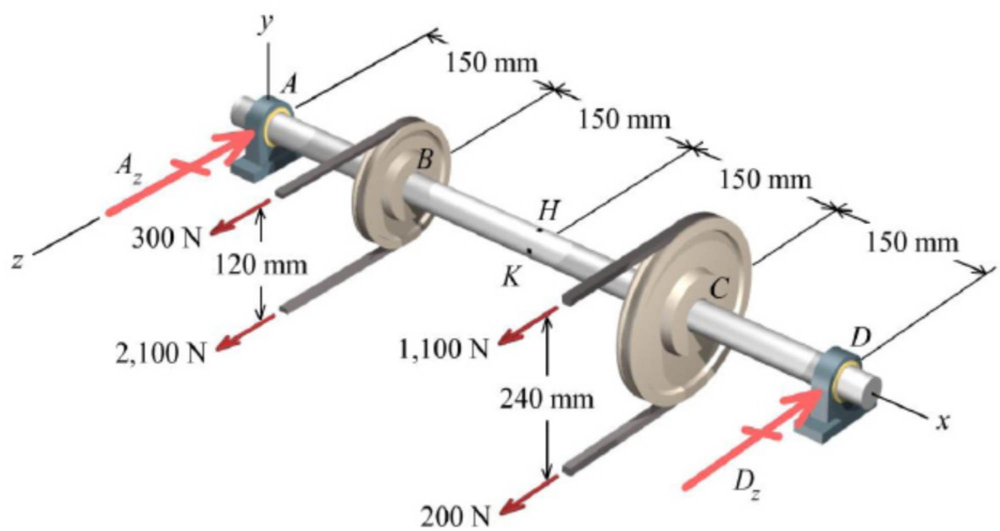
$$\tau_{max} = \frac{17.17-(-232.54)}{2} = 124.85\ MPa \qquad\qquad FS)_{Tresca} = \frac{\frac{320}{2}}{124.85} = 1.28$$

$$\sigma_M^2 = \sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 = (17.17)^2 - \big(17.17 \times (-232.54)\big) + (-232.54)^2 \rightarrow \sigma_M = 242\ MPa$$

$$FS)_{von\ Mises} = \frac{320}{242} = 1.32$$

Example 12: A steel shaft with an outside diameter of 20 mm is supported in flexible bearings at its ends. Two pulleys are keyed to the shaft, and the pulleys carry belt tensions as shown. The yield strength of the steel is 350 MPa.(a) Determine the factors of safety predicted at points *H* and *K* by the maximum-shear-stress theory of failure. (b) Determine the Mises equivalent stresses at points *H* and *K*. (c) Determine the factors of safety at points *H* and *K* predicted by the maximum-distortion-energy theory.





Equilibrium of entire shaft:

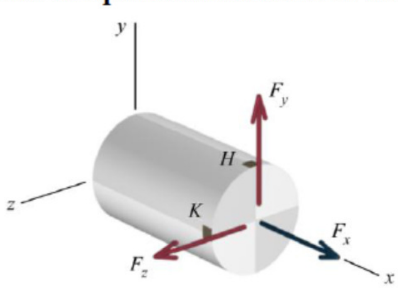
$$\Sigma F_z = -A_z - D_z + 300\text{ N} + 2{,}100\text{ N} + 1{,}100\text{ N} + 200\text{ N} = 0$$

$$\Sigma M_{A,y\text{ axis}} = -(300\text{ N})(150\text{ mm}) - (2{,}100\text{ N})(150\text{ mm})$$
$$-(1{,}100\text{ N})(450\text{ mm}) - (200\text{ N})(450\text{ mm}) + (600\text{ mm})D_z = 0$$

therefore

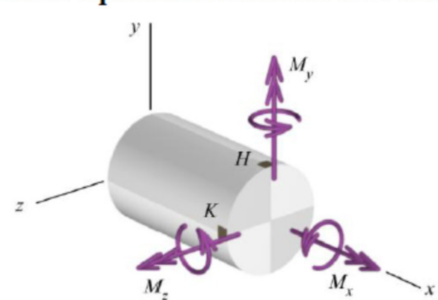
$$D_z = 1{,}575\text{ N and } A_z = 2{,}125\text{ N}$$

Detail of equivalent forces at *H* and *K*:



$$F_x = 0\text{ N}$$
$$F_y = 0\text{ N}$$
$$F_z = 1{,}100\text{ N} + 200\text{ N} - 1{,}575\text{ N}$$
$$= -275\text{ N}$$

Detail of equivalent moments at *H* and *K*:



$$M_x = (1{,}100\text{ N})(120\text{ mm}) - (200\text{ N})(120\text{ mm})$$
$$= 108{,}000\text{ N-mm}$$
$$M_y = (1{,}575\text{ N})(300\text{ mm}) - (1{,}100\text{ N})(150\text{ mm})$$
$$-(200\text{ N})(150\text{ mm}) = 277{,}500\text{ N-mm}$$
$$M_z = 0\text{ N-mm}$$

Consider point *H*.

Force F_z creates a transverse shear stress in the *xz* plane at *H*. The magnitude of this shear stress is:

$$\tau_{xz} = \frac{(275\text{ N})(666.667\text{ mm}^3)}{(7{,}853.982\text{ mm}^4)(20\text{ mm})} = 1.167\text{ MPa}$$

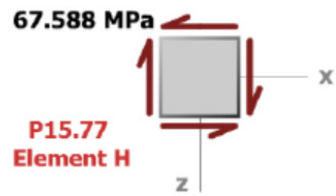
Moment M_x, which is a **torque**, creates a torsion shear stress in the *xz* plane at *H*. The magnitude of this shear stress is:

$$\tau_{xz} = \frac{M_x c}{J} = \frac{(108{,}000\text{ N-mm})(20\text{ mm}/2)}{15{,}707.963\text{ mm}^4} = 68.755\text{ MPa}$$

Moment M_y does not create bending stress at *H* because *H* is located on the neutral axis for bending about the *y* axis.

Summary of stresses at *H*:

$$\sigma_x = 0\text{ MPa}$$
$$\sigma_z = 0\text{ MPa}$$
$$\tau_{xz} = -1.167\text{ MPa} + 68.755\text{ MPa} = 67.588\text{ MPa}$$



Principal stress calculations for point *H*:

$$\sigma_{p1,p2} = \frac{(0\text{ MPa}) + (0\text{ MPa})}{2} \pm \sqrt{\left(\frac{(0\text{ MPa}) - (0\text{ MPa})}{2}\right)^2 + (-67.588\text{ MPa})^2}$$
$$= 0\text{ MPa} \pm 67.588\text{ MPa}$$

therefore, $\boxed{\sigma_{p1} = 67.588\text{ MPa}}$ and $\boxed{\sigma_{p2} = -67.588\text{ MPa}}$

Consider point *K*.

Force F_z does not cause either a normal stress or a shear stress at *K*.

Moment M_x, which is a **torque**, creates a torsion shear stress in the *xy* plane at *K*. The magnitude of this shear stress is:

$$\tau_{xy} = \frac{M_x c}{J} = \frac{(108{,}000\text{ N-mm})(20\text{ mm}/2)}{15{,}707.963\text{ mm}^4} = 68.755\text{ MPa}$$

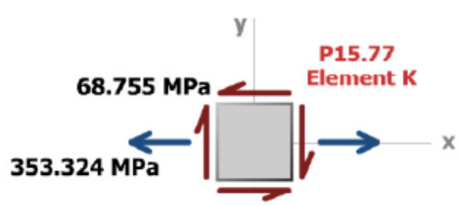
Moment M_y creates bending stress at *K*. The magnitude of this stress is:

$$\sigma_x = \frac{M_y z}{I_y} = \frac{(277{,}500\text{ N-mm})(20\text{ mm}/2)}{7{,}853.982\text{ mm}^4} = 353.324\text{ MPa}$$

Moment M_z does not create bending stress at *K* because *K* is located on the neutral axis for bending about the *z* axis.

Summary of stresses at *K*:

$$\sigma_x = 353.324\text{ MPa}$$
$$\sigma_y = 0\text{ MPa}$$
$$\tau_{xy} = -68.755\text{ MPa}$$



Principal stress calculations for point *K*:

$$\sigma_{p1,p2} = \frac{(353.324\text{ MPa}) + (0\text{ MPa})}{2} \pm \sqrt{\left(\frac{(353.324\text{ MPa}) - (0\text{ MPa})}{2}\right)^2 + (-68.755\text{ MPa})^2}$$
$$= 176.662\text{ MPa} \pm 189.570\text{ MPa}$$

therefore, $\boxed{\sigma_{p1} = 366.232\text{ MPa}}$ and $\boxed{\sigma_{p2} = -12.908\text{ MPa}}$

(a) Maximum-Shear-Stress Theory

Element H:

$$\left|\sigma_{p1} - \sigma_{p2}\right| = \left|67.588 \text{ MPa} - (-67.588 \text{ MPa})\right| = 135.176 \text{ MPa}$$

The factor of safety associated with this state of stress is:

$$\text{FS}_H = \frac{350 \text{ MPa}}{135.176 \text{ MPa}} = \boxed{2.59}$$ **Ans.**

Element K:

$$\left|\sigma_{p1} - \sigma_{p2}\right| = \left|366.232 \text{ MPa} - (-12.908 \text{ MPa})\right| = 379.140 \text{ MPa}$$

The factor of safety associated with this state of stress is:

$$\text{FS}_K = \frac{350 \text{ MPa}}{379.140 \text{ MPa}} = \boxed{0.923}$$ **Ans.**

(b) Mises equivalent stresses at points *H* and *K*:

Element H:

$$\sigma_{M,H} = \left[\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2\right]^{1/2}$$

$$= \left[(67.588 \text{ MPa})^2 - (67.588 \text{ MPa})(-67.588 \text{ MPa}) + (-67.588 \text{ MPa})^2\right]^{1/2}$$

$$= 117.066 \text{ MPa} = \boxed{117.1 \text{ MPa}}$$ **Ans.**

Element K:

$$\sigma_{M,K} = \left[\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2\right]^{1/2}$$

$$= \left[(366.232 \text{ MPa})^2 - (366.232 \text{ MPa})(-12.908 \text{ MPa}) + (-12.908 \text{ MPa})^2\right]^{1/2}$$

$$= 372.853 \text{ MPa} = \boxed{373 \text{ MPa}}$$ **Ans.**

(c) Maximum-Distortion-Energy Theory:

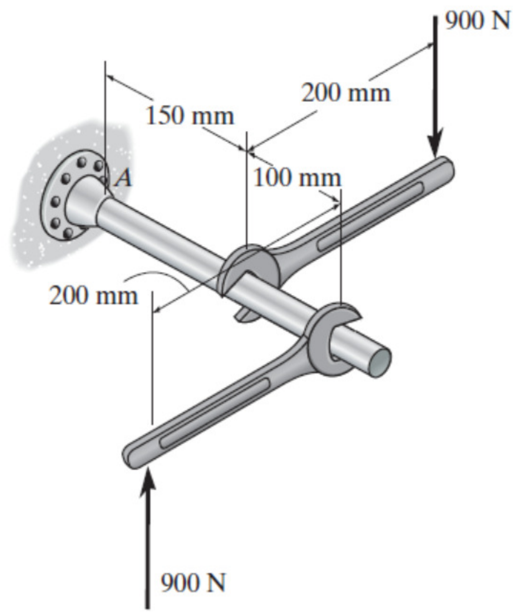
Element H:

$$\text{FS}_H = \frac{350 \text{ MPa}}{117.066 \text{ MPa}} = \boxed{2.99}$$ **Ans.**

Element K:

$$\text{FS}_K = \frac{350 \text{ MPa}}{372.853 \text{ MPa}} = \boxed{0.939}$$ **Ans.**

TBR 6: If the A-36 steel (σ_Y = 250 MPa) pipe has an outer and inner diameter of 30 mm and 20 mm, respectively, determine the factor of safety against yielding of the material at point *A* according to the maximum-distortion-energy theory.



Internal Loadings: Considering the equilibrium of the free - body diagram of the pipe's right cut segment Fig. *a*,

$\Sigma F_y = 0;\quad V_y + 900 - 900 = 0 \qquad\qquad V_y = 0$

$\Sigma M_x = 0;\ T + 900(0.4) = 0 \qquad\qquad T = -360\,\text{N}\cdot\text{m}$

$\Sigma M_z = 0;\ M_z + 900(0.15) - 900(0.25) = 0\ \ M_z = 90\,\text{N}\cdot\text{m}$

Section Properties. The moment of inertia about the z axis and the polar moment of inertia of the pipe's cross section are

$$I_z = \frac{\pi}{4}\left(0.015^4 - 0.01^4\right) = 10.15625\pi\left(10^{-9}\right)\,\text{m}^4$$

$$J = \frac{\pi}{2}\left(0.015^4 - 0.01^4\right) = 20.3125\pi\left(10^{-9}\right)\,\text{m}^4$$

Normal Stress and Shear Stress. The normal stress is caused by bending stress. Thus,

$$\sigma_Y = -\frac{My_A}{I_z} = -\frac{90(0.015)}{10.15625\pi\left(10^{-9}\right)} = -42.31\text{MPa}$$

The shear stress is caused by torsional stress.

$$\tau = \frac{Tc}{J} = \frac{360(0.015)}{20.3125\pi\left(10^{-9}\right)} = 84.62\text{ MPa}$$

In - Plane Principal Stress. $\sigma_x = -42.31$ MPa, $\sigma_z = 0$ and $\tau_{xz} = 84.62$ MPa. We have

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}{}^2}$$

$$= \frac{-42.31 + 0}{2} \pm \sqrt{\left(\frac{-42.31 - 0}{2}\right)^2 + 84.62^2}$$

$$= (-21.16 \pm 87.23)\text{ MPa}$$

Maximum Distortion Energy Theory.

$$\sigma_1{}^2 - \sigma_1\sigma_2 + \sigma_2{}^2 = \sigma_{\text{allow}}{}^2$$

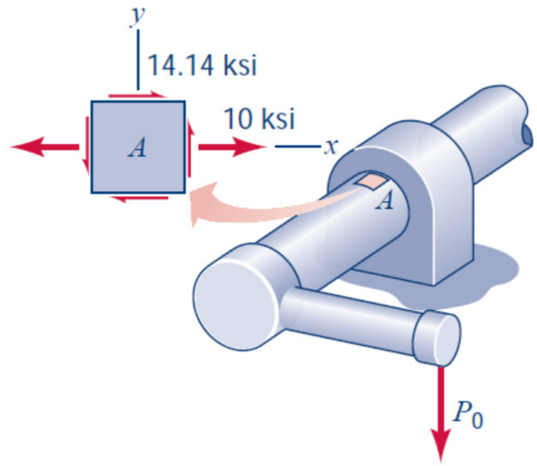
$$66.07^2 - 66.07(-108.38) + (-108.38)^2 = \sigma_{\text{allow}}{}^2$$

$$\sigma_{\text{allow}} = 152.55\text{ MPa}$$

Thus, the factor of safety is

$$F.S. = \frac{\sigma_Y}{\sigma_{\text{allow}}} = \frac{250}{152.55} = 1.64$$

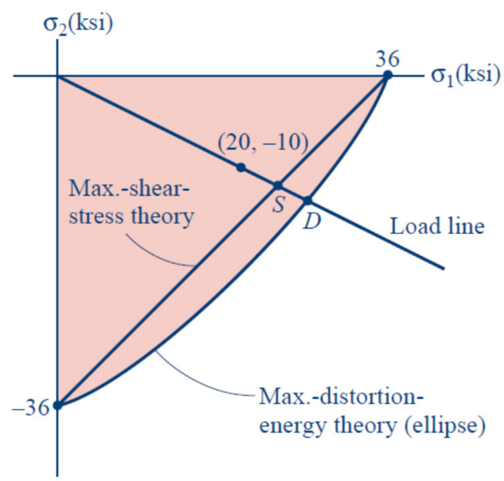
TBR 7: A force P_0 applied by a lever arm to the shaft produces stresses at the critical point A having the values shown. Determine the load $P_S = c_S\, P_0$ that would cause the shaft to fail according to the maximum-shear-stress theory, and determine the load $P_D = c_D\, P_0$ that would cause failure according to the maximum-distortion-energy theory. The shaft is made of steel with $\sigma_Y = 36$ ksi.



$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{(\frac{\sigma_x-\sigma_y}{2})^2 + {\tau_{xy}}^2}$$

$$\sigma_{1,2} = \frac{10+0}{2} \pm \sqrt{(\frac{10-0}{2})^2 + 14.14^2} \qquad \sigma_{1,2} = -10, 20\ ksi$$

* The stresses at point A are proportional to load



Tresca:

$\sigma_1 = -2\sigma_2\, , \sigma_1 - \sigma_2 = \sigma_y = 36\ ksi\, , \quad \sigma_1 = 24\ ksi \quad \sigma_2 = -12\ ksi$

$$\frac{P_s}{P_0} = \frac{24\ ksi}{20\ ksi} = 1.2 \quad P_s = 1.2\, P_0$$

$c_S = 1.2$

Von Mises:

$\sigma_1 = -2\,\sigma_2\, , {\sigma_1}^2 - \sigma_1\sigma_2 + {\sigma_2}^2 = {\sigma_y}^2 = (36\ ksi)^2$

$\sigma_1 > 0$ and $\sigma_2 < 0$

$\sigma_1 = 27.2\ ksi \quad \sigma_2 = -13.6\ ksi$

$$\frac{P_D}{P_0} = \frac{27.2\ ksi}{20\ ksi} = 1.36 \quad P_D = 1.36\, P_0$$

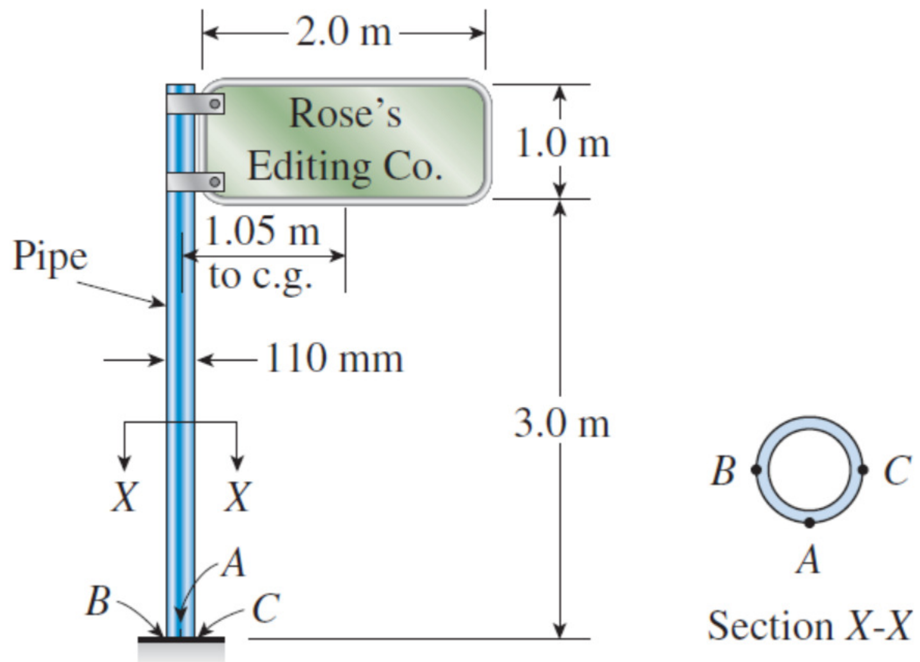
$c_D = 1.36$

OR

$$\sigma_v^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = (20)^2 - (-10\times 20) + (-10)^2 \qquad \sigma_v = 26.4\ ksi$$

$$SF = \frac{\sigma_y}{\sigma_v} = \frac{36\ ksi}{26.4\ ksi} = 1.36$$

TBR 8: A sign is supported by a pipe (σ_Y = 250 MPa) having outer diameter 110 mm and inner diameter 90 mm. The dimensions of the sign are 2.0 m × 1.0 m, and its lower edge is 3.0 m above the base. Note that the center of gravity of the sign is 1.05 m from the axis of the pipe. The wind pressure against the sign is 1.5 kPa. Determine factor of safety based on Tresca and von Mises (maximum-distortion-energy theory) criterions at points A, B, and C.



PIPE: $d_2 = 110$ mm $d_1 = 90$ mm $t = 10$ mm

$I = \frac{\pi}{64}(d_2^4 - d_1^4)$ $I = 3.966 \times 10^6$ mm^4

$I_p = 2I$ $I_p = 7.933 \times 10^6$ mm^4

$Q = \frac{1}{12}(d_2^3 - d_1^3)$

$Q = 5.017 \times 10^4$ mm^3

SIGN: $A = 2\text{m}^2$

$h = \left(3 + \frac{1}{2}\right)$ m

$b = 1.05$ m

WIND PRESSURE $p_w = 1.5$ kPa

$P = p_w A$ $P = 3$ kN

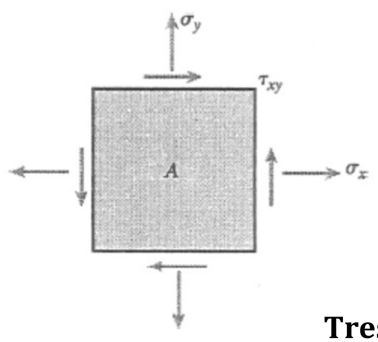
STRESS RESULTANTS AT THE BASE

$M = Ph$ $M = 10.5$ kN·m

$T = Pb$ $T = 3.15$ kN·m

$V = P$ $V = 3$ kN

MAXIMUM SHEAR STRESS AT POINT A



$\sigma_x = 0$

$\sigma_y = \frac{Md_2}{2I}$ $\sigma_y = 145.603$ MPa

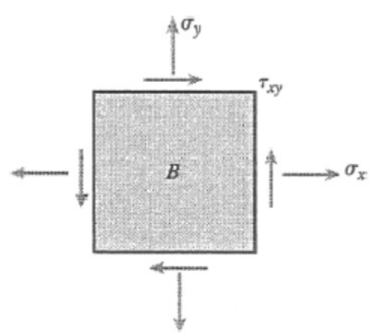
$\tau_{xy} = \frac{Td_2}{2I_p}$ $\tau_{xy} = 21.84$ MPa

$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

$\tau_{max} = 76.0$ MPa ←

Tresca: FS = (250/2)/76 = 1.64

MAXIMUM SHEAR STRESS AT POINT B



$\sigma_x = 0$

$\sigma_y = 0$

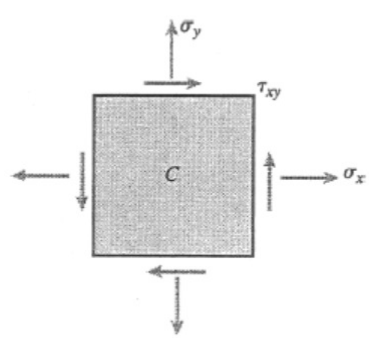
$\tau_{xy} = \frac{Td_2}{2I_p} - \frac{VQ}{I(2t)}$ $\tau_{xy} = 19.943$ MPa

$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Pure shear $\tau_{max} = 19.94$ MPa ←

Tresca: FS = (250/2)/19.94 = 6.26

MAXIMUM SHEAR STRESS AT POINT C



$\sigma_x = 0$

$\sigma_y = 0$

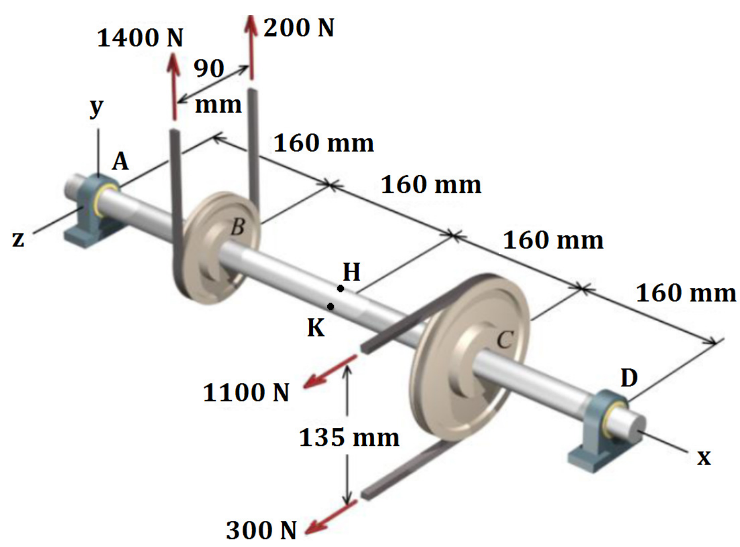
$\tau_{xy} = \frac{Td_2}{2I_p} + \frac{VQ}{I(2t)}$ $\tau_{xy} = 23.738$ MPa

$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

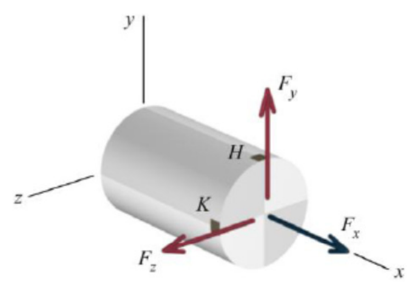
Pure shear $\tau_{max} = 23.7$ MPa ←

Tresca: FS = (250/2)/23.7 = 5.27

TBR 9: A steel shaft with an outside diameter of 20 mm is supported in flexible bearings at its ends. Two pulleys are keyed to the shaft, and the pulleys carry belt tensions as shown. The yield strength of the steel is $\sigma_Y = 350$ MPa. Determine the factors of safety predicted at points *H* and *K* by the maximum-shear-stress theory and by the maximum-distortion-energy theory.

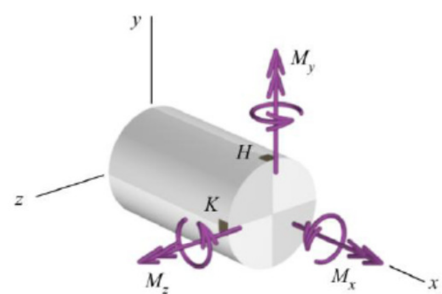


Detail of equivalent forces at *H* and *K*:



$$F_x = 0 \text{ N}$$
$$F_y = -400 \text{ N}$$
$$F_z = 1{,}100 \text{ N} + 300 \text{ N} - 1{,}050 \text{ N}$$
$$= 350 \text{ N}$$

Detail of equivalent moments at *H* and *K*:



$$M_x = (1{,}100 \text{ N})(135 \text{ mm}/2) - (300 \text{ N})(135 \text{ mm}/2)$$
$$= 54{,}000 \text{ N-mm}$$
$$M_y = (1{,}050 \text{ N})(320 \text{ mm}) - (1{,}100 \text{ N})(160 \text{ mm})$$
$$-(300 \text{ N})(160 \text{ mm}) = 112{,}000 \text{ N-mm}$$
$$M_z = -(400 \text{ N})(320 \text{ mm}) = -128{,}000 \text{ N-mm}$$

Consider point *H*.
Force F_y does not cause either a normal stress or a shear stress at *H*.

Force F_z creates a transverse shear stress in the *xz* plane at *H*. The magnitude of this shear stress is:

$$\tau_{xz} = \frac{(350 \text{ N})(666.667 \text{ mm}^3)}{(7{,}853.982 \text{ mm}^4)(20 \text{ mm})} = 1.485 \text{ MPa}$$

Moment M_x, which is a **torque**, creates a torsion shear stress in the *xz* plane at *H*. The magnitude of this shear stress is:

$$\tau_{xz} = \frac{M_x c}{J} = \frac{(54{,}000 \text{ N-mm})(20 \text{ mm}/2)}{15{,}707.963 \text{ mm}^4} = 34.377 \text{ MPa}$$

Moment M_y does not create bending stress at H because H is located on the neutral axis for bending about the y axis.

Moment M_z creates bending stress at H. The magnitude of this stress is:

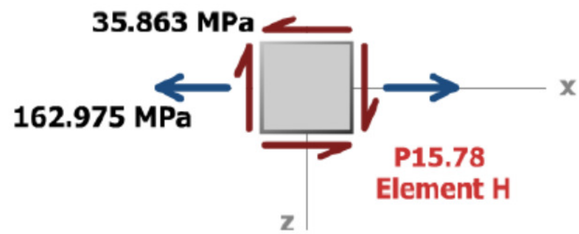
$$\sigma_x = \frac{M_z y}{I_z} = \frac{(128{,}000 \text{ N-mm})(20 \text{ mm}/2)}{7{,}853.982 \text{ mm}^4} = 162.975 \text{ MPa}$$

Summary of stresses at H:

$$\sigma_x = 162.975 \text{ MPa}$$

$$\sigma_z = 0 \text{ MPa}$$

$$\tau_{xz} = 1.485 \text{ MPa} + 34.377 \text{ MPa} = 35.863 \text{ MPa}$$



Principal stress calculations for point H:

$$\sigma_{p1,p2} = \frac{(162.975 \text{ MPa}) + (0 \text{ MPa})}{2} \pm \sqrt{\left(\frac{(162.975 \text{ MPa}) - (0 \text{ MPa})}{2}\right)^2 + (-35.863 \text{ MPa})^2}$$

$$= 81.487 \text{ MPa} \pm 89.030 \text{ MPa}$$

therefore, $\boxed{\sigma_{p1} = 170.517 \text{ MPa}}$ and $\boxed{\sigma_{p2} = -7.543 \text{ MPa}}$

(a) Maximum-Shear-Stress Theory

Element H:

$$\left|\sigma_{p1} - \sigma_{p2}\right| = \left|170.517 \text{ MPa} - (-7.543 \text{ MPa})\right| = 178.060 \text{ MPa}$$

The factor of safety associated with this state of stress is:

$$\text{FS}_H = \frac{350 \text{ MPa}}{178.060 \text{ MPa}} = \boxed{1.966}$$

(b) Mises equivalent stresses at points H

Element H:

$$\sigma_{M,H} = \left[\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2\right]^{1/2}$$

$$= \left[(170.517 \text{ MPa})^2 - (170.517 \text{ MPa})(-7.543 \text{ MPa}) + (-7.543 \text{ MPa})^2\right]^{1/2}$$

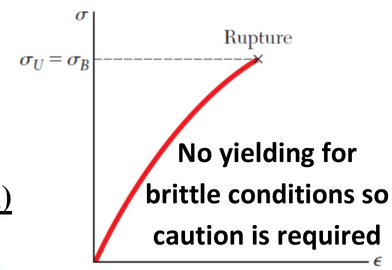
$$= 174.411 \text{ MPa} = \boxed{174.4 \text{ MPa}}$$

$$\text{FS}_H = \frac{350 \text{ MPa}}{174.411 \text{ MPa}} = \boxed{2.01}$$

Failure (Fracture) Criteria for Brittle Materials under Plane Stress

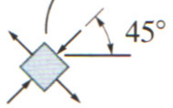
For uniaxial tensile test failure occurs when $\sigma = \sigma_U$ (ultimate stress).

For plane stress conditions a criterion needs to be defined

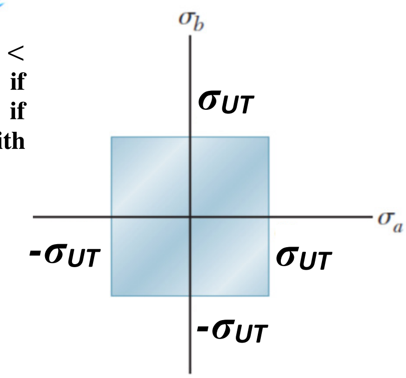


Stress-strain diagram for a typical brittle material.

Maximum-Normal-Stress Criterion (Coulomb's or Rankine; Criterion)



For Brittle materials: 1- fracture is due to normal stress and 2- $\sigma_{UT} < \sigma_{UC}$ (e.g., for cast iron $\sigma_{UC} = 4 \times \sigma_{UT}$). Shortcoming: Not good if $\sigma_a, \sigma_b < 0$. But it shows good agreement with experimental tests if there exists a tensile principal stress or for a brittle material with $\sigma_{UT} = \sigma_{UC}$ (this is seldom due to presence of cracks).

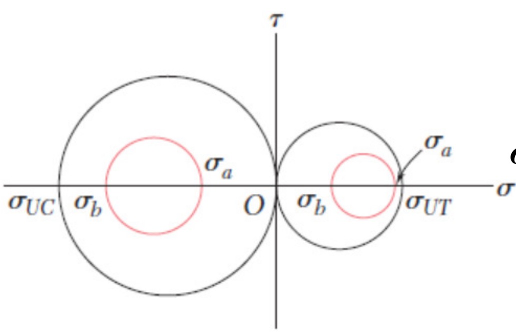


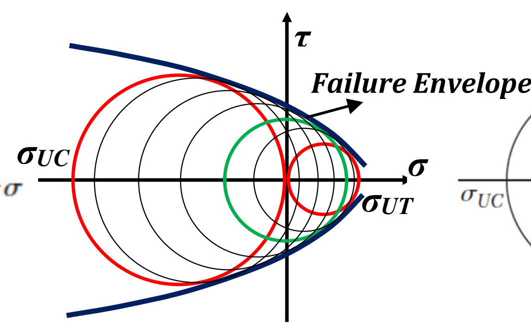
Coulomb's criterion.

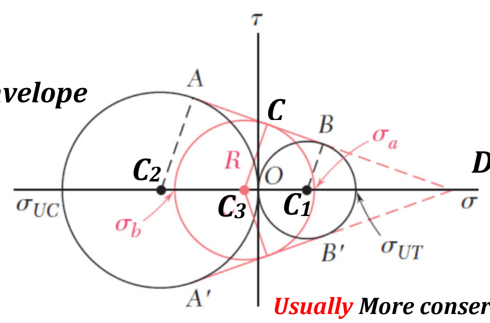
$$|\sigma_a| < \sigma_{UT} \qquad |\sigma_b| < \sigma_{UT} \qquad FS = {}^{\sigma_{UT}}\!/_{\max\,(\sigma_a, \sigma_b)}$$

Mohr's and simplifies Mohr (Coulomb-Mohr) Criterion

Applicable when results of various tests on material are available. If both principal stresses are positive, the state of stress is safe as long as $\sigma_a<\sigma_{UT}$ and $\sigma_b<\sigma_{UT}$; if both principal stresses are negative, the state of stress is safe as long as $|\sigma_a|<|\sigma_{UC}|$ and $|\sigma_b|<|\sigma_{UC}|$.

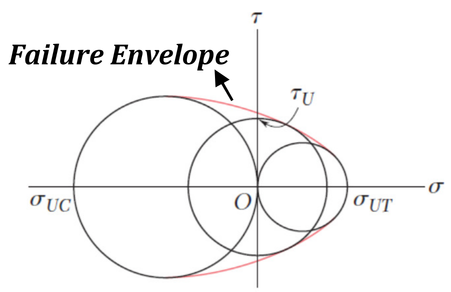






Usually More conservative than Mohr's criterion

Simplified Mohr's criterion.



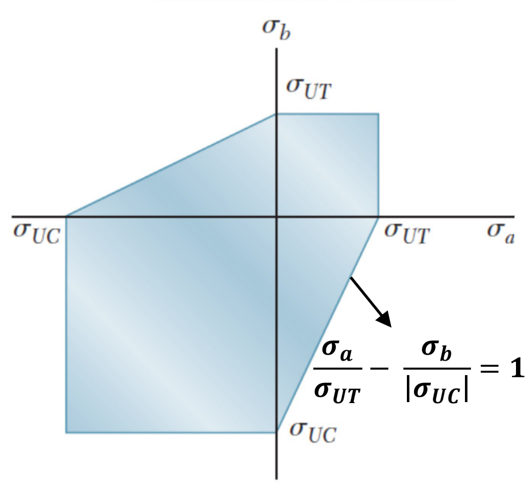
Mohr's criterion.

$$\frac{CC_3 - BC_1}{DC_3 - DC_1} = \frac{AC_2 - BC_1}{DC_2 - DC_1}$$

$$\frac{\frac{\sigma_a - \sigma_b}{2} - \frac{\sigma_{UT}}{2}}{\frac{\sigma_{UT}}{2} - \frac{\sigma_a + \sigma_b}{2}} = \frac{\frac{|\sigma_{UC}|}{2} - \frac{\sigma_{UT}}{2}}{\frac{\sigma_{UT}}{2} + \frac{|\sigma_{UC}|}{2}}$$

$$\frac{\sigma_a}{\sigma_{UT}} - \frac{\sigma_b}{|\sigma_{UC}|} = 1$$

Only when principal stresses have opposite sign

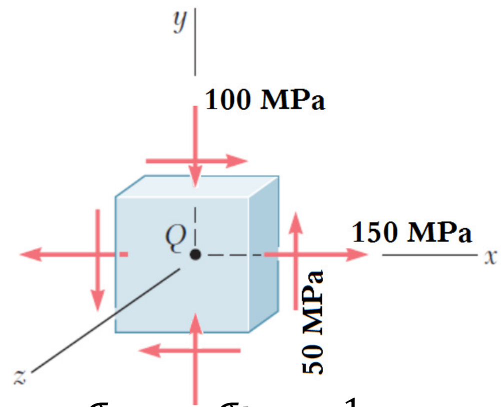


For design, incorporating the factor of safety FS, divide all strengths by FS:

$$\frac{\sigma_a}{\sigma_{UT}} - \frac{\sigma_b}{|\sigma_{UC}|} = \frac{1}{FS},$$

If principal stresses have the same sign: $FS = {}^{\sigma_{UT}}\!/_{\max\,(\sigma_1, \sigma_2)}$ ***OR*** $FS = {}^{|\sigma_{UC}|}\!/_{\max\,(|\sigma_1|, |\sigma_2|)}$

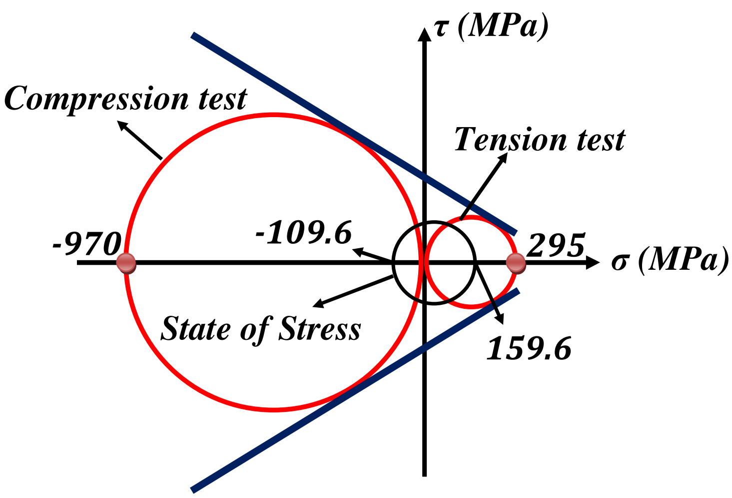
Mohr's criteria can be used in ductile conditions in which yield stress in tension and compression are very different as Tresca and von Mises criterions both assume that yield stress in tension and compression are equal.



Example 13: For a certain point of a cast-iron machine frame the state of stress on an element is as shown. Find factor of safety based on Mohr's criterion (σ_{UT} = 295 MPa and σ_{UC} = 970 MPa).

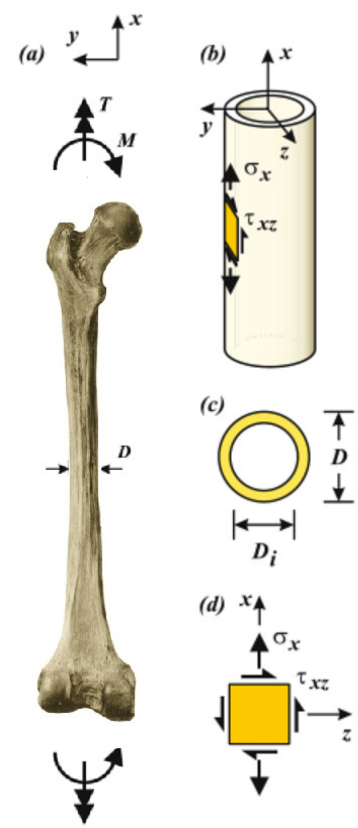
$$\sigma_{a,b} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{(\frac{\sigma_x-\sigma_y}{2})^2 + {\tau_{xy}}^2}$$

$$= \frac{150-100}{2} \pm \sqrt{(\frac{150+100}{2})^2 + 50^2} = 159.6\ MPa, -109.6\ MPa \quad \frac{\sigma_a}{\sigma_{UT}} - \frac{\sigma_b}{\sigma_{UC}} = \frac{1}{FS} \rightarrow$$



$$\frac{159.6}{295} - \frac{-109.6}{970} = \frac{1}{FS} \rightarrow FS = 1.53$$

How to calculate FS using Mohr's circle?



Example 14: The shaft of a femur can be approximated as a hollow cylindrical shaft. The loads that cause femur bones to fracture are axial torque and bending moments. During strenuous activities (e.g., skiing) the femur is subjected to torque of *T* = 100 Nm. Determine the maximal bending moment *M* that the bone can support without failure according to maximum normal stress criteria (*D* = 24 mm, D_i = 16 mm, σ_{ut} = 120 MPa, σ_{uc} = 240 MPa).

$$\tau_{xz} = \frac{Tr}{J} = \frac{100\,000\ (Nmm)\ (12\ mm)}{\frac{\pi}{2}\ (12^4 - 8^4)\ mm^4} = 45.9\ MPa$$

$$\sigma_x = \frac{Mr}{I} = \frac{M\ (12\ mm)}{\frac{\pi}{4}\ (12^4 - 8^4)\ mm^4} = 0.000918\ M$$

$$\sigma_z = 0;\ \ \sigma_{a,b} = \frac{\sigma_x+\sigma_z}{2} \pm \sqrt{(\frac{\sigma_x-\sigma_z}{2})^2 + {\tau_{xz}}^2} =$$

$$0.000459\ M \pm \sqrt{(0.000459\ M)^2 + (45.9)^2}$$

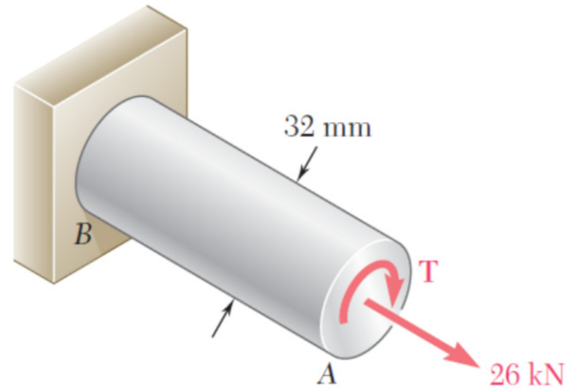
$$\sigma_{max} = 0.000459\ M + \sqrt{(0.000459\ M)^2 + (45.9)^2} = 120\ MPa \ \rightarrow\ \boldsymbol{M = 111594\ Nmm}$$

Based on Coulomb-Mohr:

$$\frac{0.000459\ M + \sqrt{(0.000459\ M)^2 + (45.9)^2}}{120} - \frac{0.000459\ M - \sqrt{(0.000459\ M)^2 + (45.9)^2}}{240} = 1$$

$$\boldsymbol{M = 99555\ Nmm} \qquad (\textit{Which criteria is more conservative? Why?})$$

Example 15: The cast-aluminum rod shown is made of an alloy for which σ_{UT} = 60 MPa and σ_{UC} = 120 MPa. Using Mohr's criterion, determine the magnitude of the torque T for which failure should be expected.



$$\sigma_x = \frac{P}{A} = \frac{26\,000\ N}{\frac{\pi}{4}(32\ mm)^2} = 32.3\ MPa$$

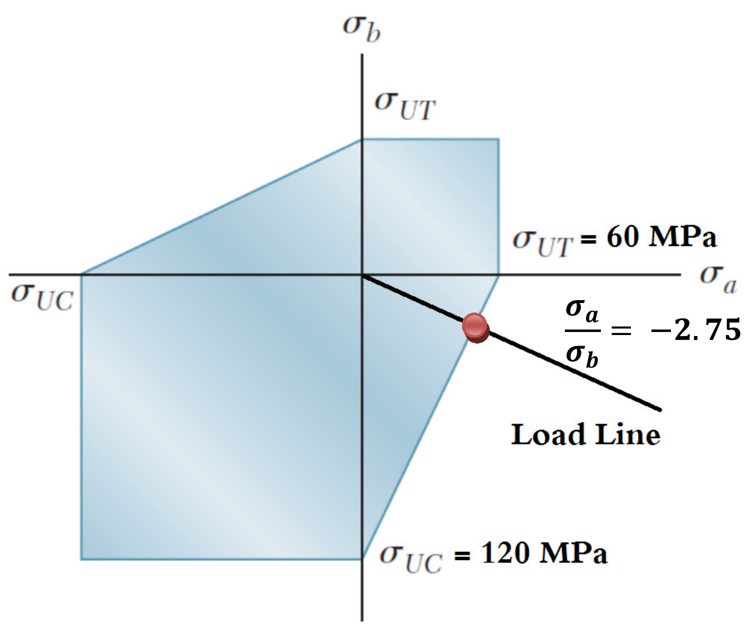
$$\tau_{xy} = \frac{Tr}{J} = \frac{T\ (16\ mm)}{\frac{\pi}{2}\ (16\ mm)^4} = 1.55 \times 10^{-4}\ T$$

$$\sigma_{a,b} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{(\frac{\sigma_x-\sigma_y}{2})^2 + {\tau_{xy}}^2}$$

$$\sigma_a = \frac{\sigma_x}{2} + \sqrt{(\frac{\sigma_x}{2})^2 + {\tau_{xy}}^2}$$

$$\sigma_b = \frac{\sigma_x}{2} - \sqrt{(\frac{\sigma_x}{2})^2 + {\tau_{xy}}^2}$$

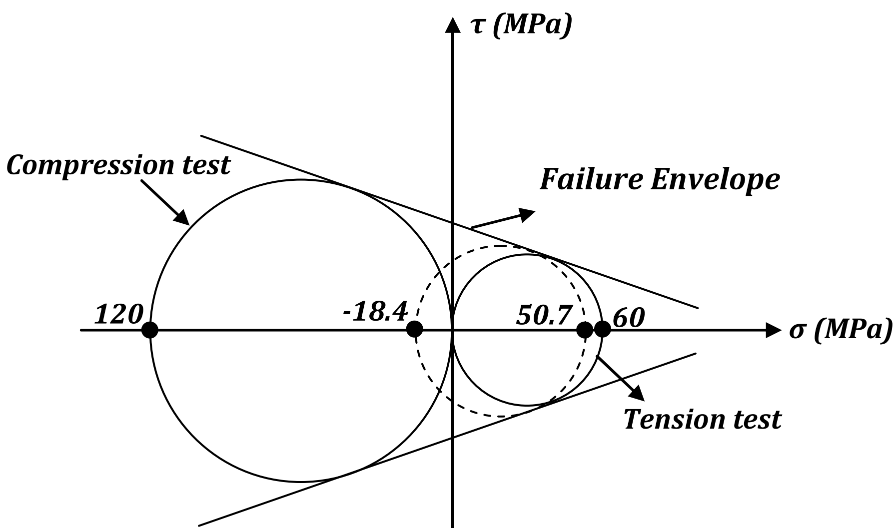
$$\sigma_a > 0 \ and\ \sigma_b < 0$$



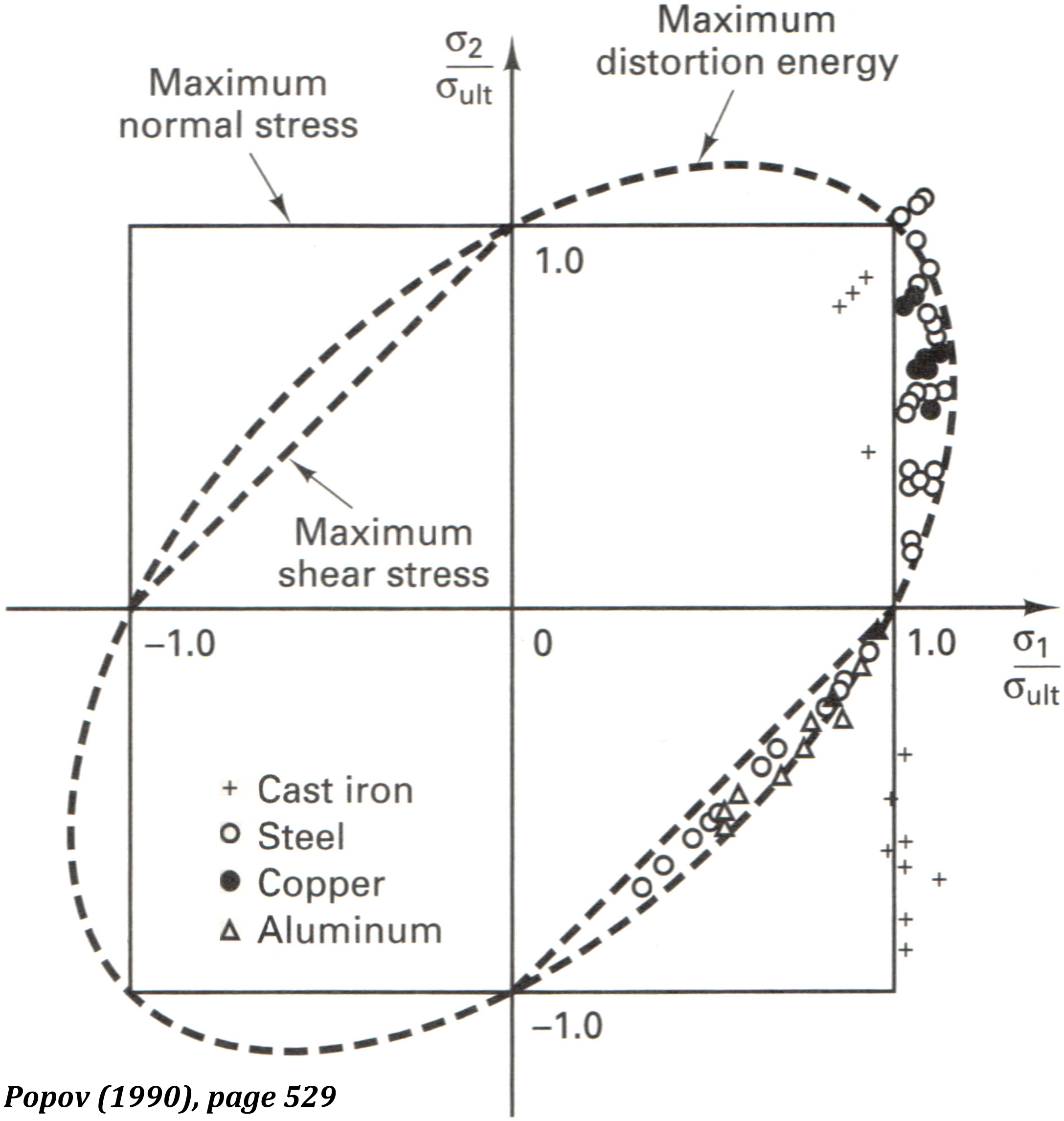
$$\frac{\sigma_a}{\sigma_{UT}} - \frac{\sigma_b}{\sigma_{UC}} = 1$$

$$\frac{16.15\ MPa+\sqrt{(16.15\ MPa)^2+(1.55\times10^{-4}\ T)^2}}{60\ MPa} - \frac{16.15\ MPa-\sqrt{(16.15\ MPa)^2+(1.55\times10^{-4}\ T)^2}}{120\ MPa} = 1$$

$T = 196900\ Nmm = 196.9\ Nm \quad \sigma_a = 50.7\ MPa \quad \sigma_b = -18.4\ MPa$



Comparison of Yield and Fracture Criteria with Test Data



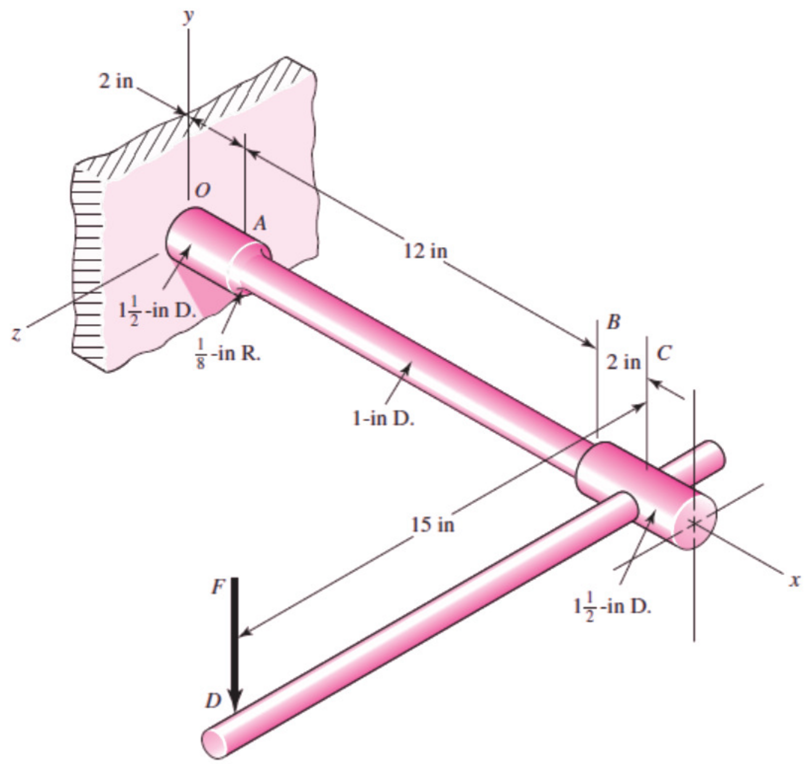
E.P. Popov (1990), page 529

<u>Remark: Pure shear test</u>

Principal stress: $\sigma_a = \tau_u \;\; and \; \sigma_b = -\tau_u \;\; \rightarrow \;\; \frac{\sigma_a}{\sigma_{UT}} - \frac{\sigma_b}{\sigma_{UC}} = 1 \;\rightarrow\; \frac{\tau_u}{\sigma_{UT}} - \frac{-\tau_u}{\sigma_{UC}} = 1$

$$\tau_u = \frac{\sigma_{UT}\sigma_{UC}}{\sigma_{UT} + \sigma_{UC}} \quad if \; \sigma_{UT} = \sigma_{UC} \quad \rightarrow \quad \tau_u = \frac{\sigma_{UT}}{2}$$

TBR 10: The cast-aluminum rod shown is made of an alloy for which σ_{UT} = 31000 psi and σ_{UC} = 109000 psi. Using Mohr's criterion, determine the maximum magnitude of the force F for which factor of safety is equal to 2 at point A (neglect shear stress due to *F* and only consider bending and torsion stresses at point A).



$$\sigma_x = \frac{Mr}{I} = \frac{(14F)(0.5\ in)}{\frac{\pi}{4}\ (0.5\ in)^4} = 142.6F$$

$$\tau_{xy} = \frac{Tr}{J} = \frac{(15F)(0.5\ in)}{\frac{\pi}{2}(0.5\ in)^4} = 76.4\ F$$

$$\sigma_{a,b} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{(\frac{\sigma_x-\sigma_y}{2})^2 + \tau_{xy}{}^2}$$

$$\sigma_a = \frac{\sigma_x}{2} + \sqrt{(\frac{\sigma_x}{2})^2 + \tau_{xy}{}^2} = 175.8\ F$$

$$\sigma_b = \frac{\sigma_x}{2} - \sqrt{(\frac{\sigma_x}{2})^2 + \tau_{xy}{}^2} = \ -33.2\ F$$

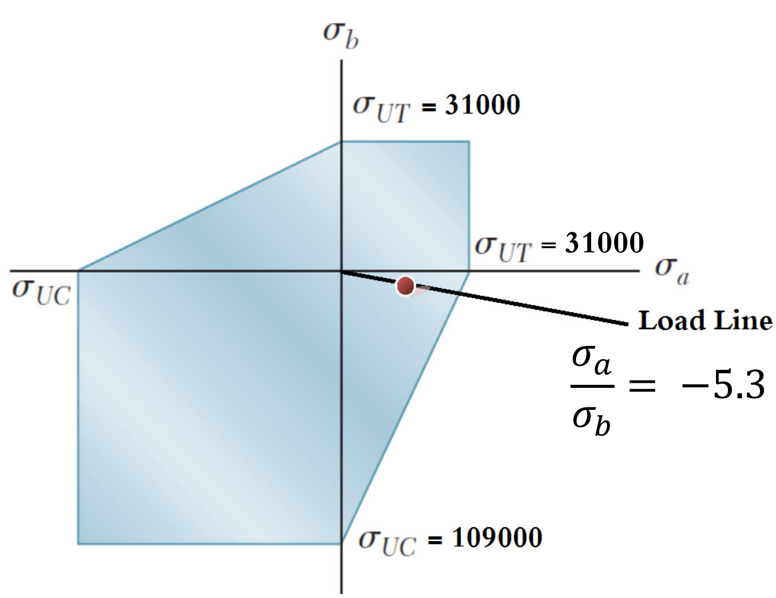
$$\frac{\sigma_a}{\sigma_{UT}} - \frac{\sigma_b}{\sigma_{UC}} = \frac{1}{2}$$

$$\frac{175.8F}{31000} - \frac{-33.2F}{109000} = \frac{1}{2}$$

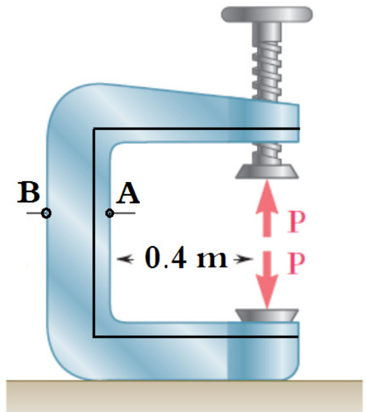
$$F = 83.5\ lbf$$

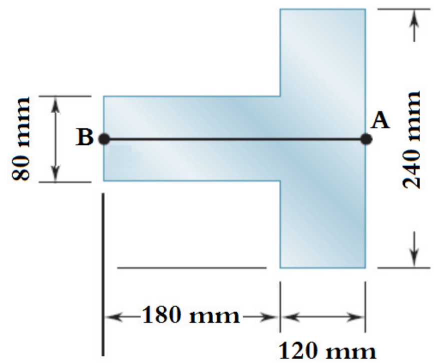
$$\sigma_a = 14680\ psi$$

$$\sigma_b = 2772\ psi$$

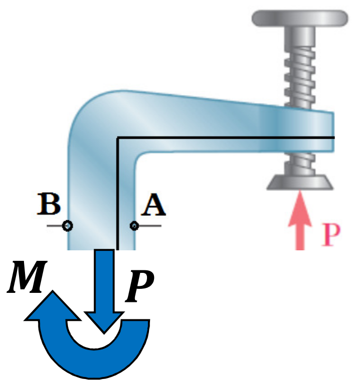


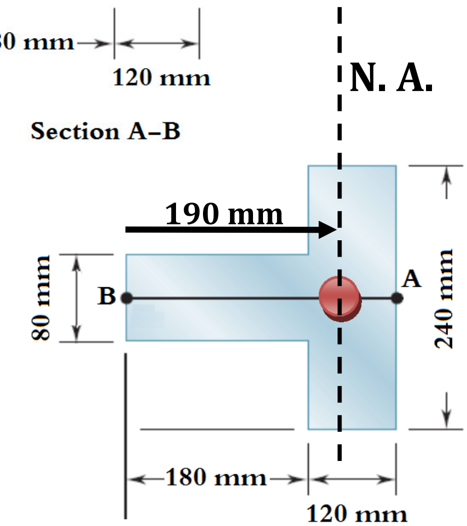
TBR 11: The shown press is made of cast iron having ultimate strength in tension (σ_{UT} = 170 MPa) and compression (σ_{UC} = 650 MPa). Calculate the allowable load P according to the Mohr-Coulomb criteria and based on a factor of safety equal to 2.5.





Section A–B





Section A–B

$$\bar{y} = \frac{\sum yA}{\sum A} = \frac{90 \times (180 \times 80) + 240 \times (120 \times 240)}{180 \times 80 + 120 \times 240} = 190\ mm$$

$$M = P(400 + 180 + 120 - 190) = 510\ P\ (Nmm)$$

$$I_{N.A.} = \frac{1}{12}80(180)^3 + (80 \times 180)(100)^2 + \frac{1}{12}240(120)^3 + (120 \times 240)(50)^2 = 2.8944 \times 10^8\ mm^4$$

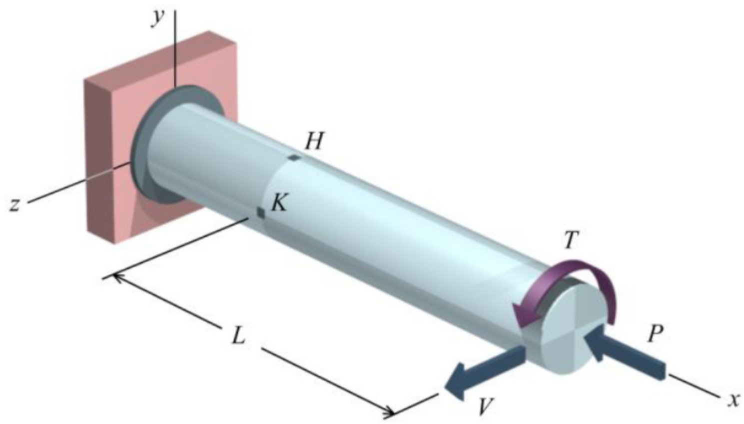
$$\sigma_{\max)_{Tensile}} = \frac{P}{A} + \frac{Mc_{max}}{I} = \frac{P}{(180 \times 80 + 120 \times 240)} + \frac{(510\ P)(110\ mm)}{2.8944 \times 10^8\ mm^4} = 0.000217\ P$$

$$\sigma_{\max)_{Comp}} = \frac{P}{A} + \frac{Mc_{max}}{I} = \frac{P}{(180 \times 80 + 120 \times 240)} - \frac{(510\ P)(190\ mm)}{2.8944 \times 10^8\ mm^4} = -0.000312\ P$$

$$\frac{\sigma_a}{\sigma_{UT}} - \frac{\sigma_b}{\sigma_{UC}} = \frac{1}{2.5} \rightarrow \frac{0}{170\ MPa} - \frac{-0.000312\ P}{650\ MPa} = \frac{1}{2.5} \rightarrow P = 834306\ N = 834.3\ kN$$

$$\frac{\sigma_a}{\sigma_{UT}} - \frac{\sigma_b}{\sigma_{UC}} = \frac{1}{2.5} \rightarrow \frac{0.000217}{170\ MPa} - \frac{0}{650\ MPa} = \frac{1}{2.5} \rightarrow P = 313406\ N = 313.4\ kN\ (controls)$$

TBR 12: A 1.25-in.-diameter solid shaft is subjected to an axial force of P = 7,000 lb, a horizontal force of V = 1,400 lb, and a concentrated torque of T = 220 lb-ft, acting in the directions shown. Assume L = 6.0 in. The ultimate failure strengths for this material are 36 ksi in tension and 50 ksi in compression. Use the Mohr failure criterion to evaluate the safety of this component at points H and K.



$$A = \frac{\pi}{4}(1.25 \text{ in.})^2 = 1.227185 \text{ in.}^2 \qquad J = \frac{\pi}{32}(1.25 \text{ in.})^4 = 0.239684 \text{ in.}^4$$

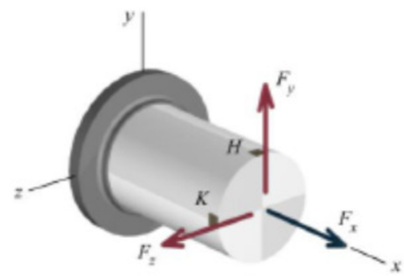
$$Q = \frac{(1.25 \text{ in.})^3}{12} = 0.162760 \text{ in.}^3 \qquad I_y = I_z = \frac{\pi}{64}(1.25 \text{ in.})^4 = 0.119842 \text{ in.}^4$$

Equivalent forces at *H* and *K*:

$F_x = -7{,}000$ lb

$F_y = 0$ lb

$F_z = 1{,}400$ lb



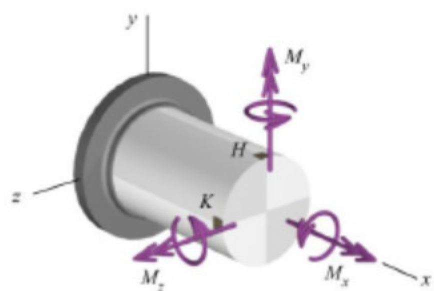
Equivalent moments at *H* and *K*:

$M_x = 220 \text{ lb-ft} = 2{,}640 \text{ lb-in.}$

$M_y = -(1{,}400 \text{ lb})(6 \text{ in.}) = -8{,}400 \text{ lb-in.}$

$M_z = 0 \text{ lb-in.}$

Each of the non-zero forces and moments will be evaluated to determine whether stresses are created at the point of interest.



Consider point *H*.

Force F_x creates an axial stress at H. The magnitude of this normal stress is:

$$\sigma_x = \frac{7{,}000 \text{ lb}}{1.227185 \text{ in.}^2} = 5{,}704.113 \text{ psi}$$

Force F_z creates a transverse shear stress in the xz plane at H. The magnitude of this shear stress is:

$$\tau_{xz} = \frac{(1{,}400 \text{ lb})(0.162760 \text{ in.}^3)}{(0.119842 \text{ in.}^4)(1.25 \text{ in.})} = 1{,}521.097 \text{ psi}$$

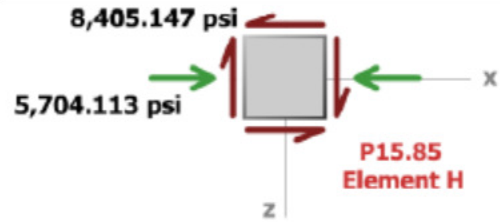
Moment M_x, which is a **torque**, creates a torsion shear stress in the *xz* plane at *H*. The magnitude of this shear stress is:

$$\tau_{xz} = \frac{M_x c}{J} = \frac{(2{,}640 \text{ lb-in.})(1.25 \text{ in./2})}{0.239684 \text{ in.}^4} = 6{,}884.050 \text{ psi}$$

Moment M_y does not create bending stress at *H* because *H* is located on the neutral axis for bending about the *y* axis.

Summary of stresses at *H*:

$$\sigma_x = -5{,}704.113 \text{ psi}$$
$$\sigma_z = 0 \text{ psi}$$
$$\tau_{xz} = 1{,}521.097 \text{ psi} + 6{,}884.050 \text{ psi} = 8{,}405.147 \text{ psi}$$



Principal stress calculations for point *H*:

$$\sigma_{p1,p2} = \frac{(-5{,}704.113 \text{ psi}) + (0 \text{ psi})}{2} \pm \sqrt{\left(\frac{(-5{,}704.113 \text{ psi}) - (0 \text{ psi})}{2}\right)^2 + (-8{,}405.147 \text{ psi})^2}$$
$$= -2{,}852.057 \text{ psi} \pm 8{,}875.850 \text{ psi}$$

therefore, $\boxed{\sigma_{p1} = 6{,}023.794 \text{ psi}}$ and $\boxed{\sigma_{p2} = -11{,}727.907 \text{ psi}}$

Mohr failure criterion at point *H*:

$$\frac{\sigma_{p1}}{\sigma_{UT}} - \frac{\sigma_{p2}}{\sigma_{UC}} = \frac{6{,}023.794 \text{ psi}}{36{,}000 \text{ psi}} - \frac{-11{,}727.907 \text{ psi}}{50{,}000 \text{ psi}}$$
$$= 0.167 - (-0.235)$$
$$= \boxed{0.402 \qquad \therefore \text{ acceptable}}$$

Ans.

Consider point *K*.
Force F_x creates an axial stress at *K*. The magnitude of this normal stress is:

$$\sigma_x = \frac{7{,}000 \text{ lb}}{1.227185 \text{ in.}^2} = 5{,}704.113 \text{ psi}$$

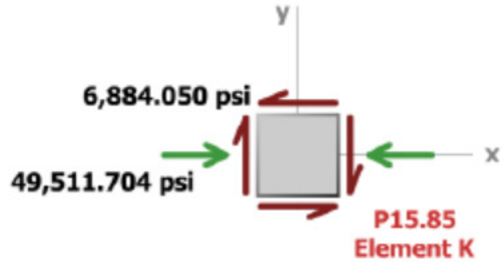
Force F_z does not cause either a normal stress or a shear stress at *K*.

Moment M_x, which is a **torque**, creates a torsion shear stress in the *xy* plane at *K*. The magnitude of this shear stress is:

$$\tau_{xy} = \frac{M_x c}{J} = \frac{(2{,}640 \text{ lb-in.})(1.25 \text{ in./2})}{0.239684 \text{ in.}^4} = 6{,}884.050 \text{ psi}$$

Moment M_y creates bending stress at *K*. The magnitude of this stress is:

$$\sigma_x = \frac{M_y z}{I_y} = \frac{(8{,}400 \text{ lb-in.})(1.25 \text{ in./2})}{0.119842 \text{ in.}^4} = 43{,}807.591 \text{ psi}$$



Summary of stresses at K:

$$\sigma_x = -5,704.113 \text{ psi} - 43,807.591 \text{ psi} = -49,511.704 \text{ psi}$$

$$\sigma_y = 0 \text{ psi}$$

$$\tau_{xy} = -6,884.050 \text{ psi}$$

Principal stress calculations for point K:

$$\sigma_{p1,p2} = \frac{(-49,511.704 \text{ psi}) + (0 \text{ psi})}{2} \pm \sqrt{\left(\frac{(-49,511.704) - (0 \text{ psi})}{2}\right)^2 + (-6,884.050 \text{ psi})^2}$$

$$= -24,755.852 \text{ psi} \pm 25,695.182 \text{ psi}$$

therefore, $\boxed{\sigma_{p1} = 939.330 \text{ psi}}$ and $\boxed{\sigma_{p2} = -50,451.034 \text{ psi}}$

Mohr failure criterion at point K:

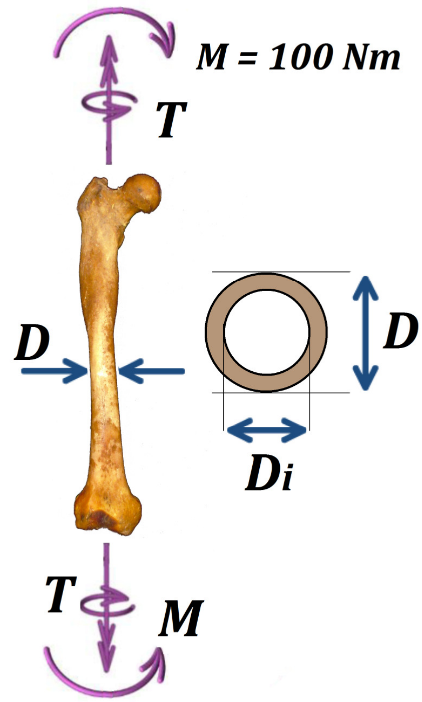
$$\frac{\sigma_{p1}}{\sigma_{UT}} - \frac{\sigma_{p2}}{\sigma_{UC}} = \frac{939.330 \text{ psi}}{36,000 \text{ psi}} - \frac{-50,451.034 \text{ psi}}{50,000 \text{ psi}}$$

$$= 0.026 - (-1.009)$$

$$= \boxed{1.035 \quad \therefore \text{ not acceptable}}$$

Ans.

TBR 13: The shaft of a femur can be approximated as a hollow cylindrical shaft. The loads that cause femur bones to fracture are axial torque and bending moments. During strenuous activities (e.g., skiing) the femur is subjected to M = 100 Nm. Determine the maximal torque T that the bone can support without failure according to Coulomb-Mohr criterion and factor of safety of 1.2 (D = 24 mm, D_i = 16 mm, σ_{ut} = 120 MPa, σ_{uc} = 240 MPa).



$$\tau_{xz} = \frac{Tr}{J} = \frac{T\ (12\ mm)}{\frac{\pi}{2}\ (12^4 - 8^4)\ mm^4} = 0.000459\ T$$

$$\sigma_x = \frac{Mr}{I} = \frac{(100\ 000\ Nmm)\ (12\ mm)}{\frac{\pi}{4}\ (12^4 - 8^4)\ mm^4} = 91.8\ MPa$$

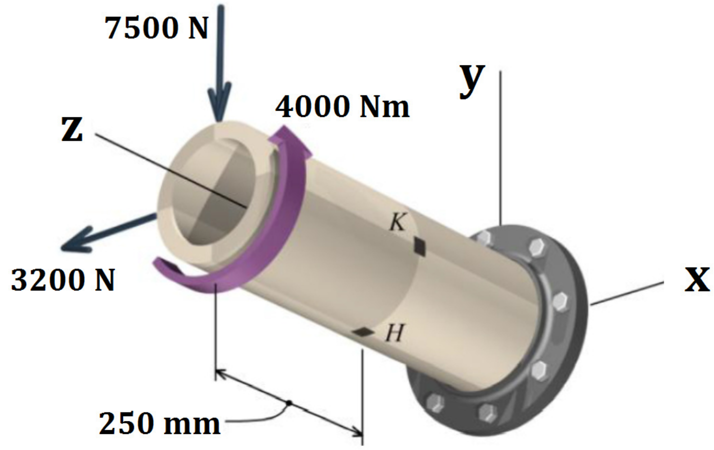
$$\sigma_z = 0;\ \ \sigma_{a,b} = \frac{\sigma_x + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + {\tau_{xz}}^2} = 45.9 \pm \sqrt{(45.9)^2 + (0.000459\ T)^2}$$

Based on Coulomb-Mohr:

$$\frac{45.9 + \sqrt{(45.9)^2 + (0.000459\ T)^2}}{120} - \frac{45.9 - \sqrt{(45.9)^2 + (0.000459\ T)^2}}{240} = \frac{1}{1.2}$$

$$\boldsymbol{T = 50237.8\ Nmm}$$

TBR 14: A pipe (σ_Y = 95 MPa) with an outside diameter of 140 mm and a wall thickness of 5 mm is subjected to the loadings shown. The internal pressure in the pipe is 1,600 kPa. (a) Determine factor of safety at point *K* according to Tresca and Mises criterion. (b) if the structure is put in the brittle conditions find factor of safety at point *H* according to Coulomb-Mohr criterion (σ_{ut} = 80 MPa, σ_{uc} = 160 MPa).



$$A = \pi\,(70^2\,mm - 65^2\,mm) = 2120.57\ mm^2$$

$$J = \frac{\pi}{2}\,(70^4 - 65^4) = 9.675\times10^6\ mm^4$$

$$I = \frac{\pi}{4}\,(70^4 - 65^4) = 4.837\times10^6\ mm^4$$

at point K

$$\tau_{zy} = \frac{Tr}{J} = \frac{4000\times10^3\ Nmm \times 70\ mm}{9.675\times10^6\ mm^4} = 28.94\ MPa$$

$$\tau_{zy} = \frac{VQ}{It} = \frac{(7500\,N)\left\{\frac{4\times70}{3\pi}\times\frac{\pi}{2}70^2 - \frac{4\times65}{3\pi}\times\frac{\pi}{2}65^2\right\} mm^3}{(4.837\times10^6\ mm^4)(5\times2\ mm)} = 7.66\ MPa$$

$$\sigma_z = \frac{(3200\,N\times250\,mm)(70\,mm)}{4.837\times10^6\ mm^4} = 11.57\ MPa$$

$$\sigma_{long} = \frac{Pr}{2t} = \frac{(1.6\,MPa)(67.5\,mm)}{2\times5\,mm} = 10.8\ MPa$$

$$\sigma_{hoop} = \frac{Pr}{t} = 21.6\ MPa$$

$$\therefore\ \tau_{zy} = 28.94\,MPa - 7.66\,MPa = 21.9\,MPa$$

$$\sigma_z = 11.57\,MPa + 10.8\,MPa = 22.37\ MPa$$

$$\sigma_y = 21.6\ MPa$$

$$\sigma_{1,2} = \frac{21.6+22.37}{2} \pm \sqrt{\left(\frac{21.6-22.37}{2}\right)^2 + 21.9^2} = 21.985 \pm 21.90 = \begin{matrix}43.88\\0.085\end{matrix}\ MPa$$

$$F.S._{Tresca} = \frac{\tau_Y}{\tau_{max}} = \frac{95/2}{\frac{43.88-0}{2}} = 2.165$$

$$F.S._{Mises} = \frac{\sigma_Y}{\sigma_{mises}} = \frac{95\ MPa}{(43.88^2 - 43.88\times0.085 + 0.085^2)^{0.5}} = 2.167$$

at Point H

$$\tau_{3x} = \frac{Tr}{J} = 28.94 \text{ MPa}$$

$$\tau_{3x} = \frac{VQ}{It} = \frac{(3200N)\left\{\frac{4\times70}{3\pi}\frac{\pi}{2}70^2 - \frac{4\times65}{3\pi}\times\frac{\pi}{2}65^2\right\} mm^3}{(4.837\times10^6\ mm^4)\ (2\times5\ mm)} = 3.01 \text{ MPa}$$

$$\sigma_3 = \frac{Mc}{I} = \frac{(7500\times250\ Nmm)(70\ mm)}{4.837\times10^6\ mm^4} = -27.13 \text{ MPa}$$

$\sigma_{long} = 10.8$ MPa , $\sigma_{hoop} = 21.6$ MPa

$\sigma_x = 21.6$ MPa

$\sigma_3 = -27.13 + 10.8 = -16.33$ MPa

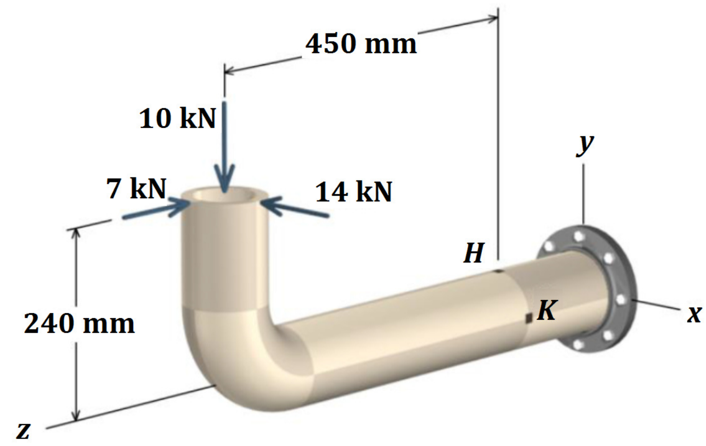
$\tau_{3x} = 28.94 - 3.01 = 25.9$ MPa

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_3}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_3}{2}\right)^2+\tau_{x3}^2} = 2.635 \pm 32.1 = \begin{matrix}34.73\\ -29.47\end{matrix} \text{ MPa}$$

$$\frac{\sigma_1}{\sigma_{ut}} - \frac{\sigma_2}{\sigma_{uc}} = \frac{1}{F_s} \rightarrow \frac{34.73}{80} - \frac{-29.47}{160} = \frac{1}{F_s}$$

$F_s = 1.617$

TBR 15: A pipe (σ_Y = 250 MPa) with an outside diameter of 95 mm and a wall thickness of 5 mm is subjected to the loadings shown. (a) Determine factor of safety at point *K* according to Tresca and Mises criterion. (b) if the structure is put in the brittle conditions find factor of safety at point *H* according to Coulomb-Mohr criterion (σ_{ut} = 250 MPa, σ_{uc} = 400 MPa).



Solution

Section properties:

$$A = \frac{\pi}{4}\left[(95\text{ mm})^2 - (85\text{ mm})^2\right] = 1,413.717\text{ mm}^2$$

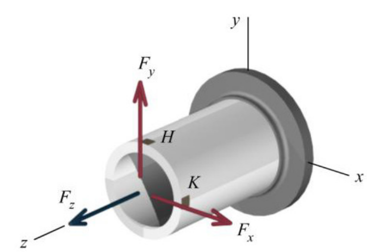
$$J = \frac{\pi}{32}\left[(95\text{ mm})^4 - (85\text{ mm})^4\right] = 2,871,612.035\text{ mm}^4$$

$$I_x = I_y = \frac{\pi}{64}\left[(95\text{ mm})^4 - (85\text{ mm})^4\right] = 1,435,806.017\text{ mm}^4$$

$$Q = \frac{1}{12}\left[(95\text{ mm})^3 - (85\text{ mm})^3\right] = 20,270.833\text{ mm}^3$$

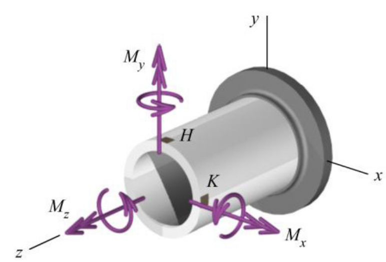
Equivalent forces at *H* and *K*:

$$F_x = -14\text{ kN} = -14,000\text{ N}$$
$$F_y = -10\text{ kN} = -10,000\text{ N}$$
$$F_z = -7\text{ kN} = -7,000\text{ N}$$



Equivalent moments at *H* and *K*:

$$M_x = (10\text{ kN})(450\text{ mm}) - (7\text{ kN})(240\text{ mm})$$
$$= 2,820\text{ kN-mm} = 2.820\times10^6\text{ N-mm}$$
$$M_y = -(14\text{ kN})(450\text{ mm})$$
$$= -6,300\text{ kN-mm} = -6.300\times10^6\text{ N-mm}$$
$$M_z = (14\text{ kN})(240\text{ mm})$$
$$= 3,360\text{ kN-mm} = 3.36\times10^6\text{ N-mm}$$



Each of the non-zero forces and moments will be evaluated to determine whether stresses are created at the point of interest.

(41)

(a) Consider point *K*.
Force F_x does not cause either a normal stress or a shear stress at *K*.

Force F_y creates a transverse shear stress in the *yz* plane at *K*. The magnitude of this shear stress is:

$$\tau_{yz} = \frac{(10{,}000\text{ N})(20{,}270.833\text{ mm}^3)}{(1{,}435{,}806.017\text{ mm}^4)\left[(95\text{ mm})-(85\text{ mm})\right]} = 14.118\text{ MPa}$$

Force F_z creates an axial stress at *K*. The magnitude of this normal stress is:

$$\sigma_z = \frac{7{,}000\text{ N}}{1{,}413.717\text{ mm}^2} = 4.951\text{ MPa}$$

Moment M_x does not create bending stress at *K* because *K* is located on the neutral axis for bending about the *x* axis.

Moment M_y creates bending stress at *K*. The magnitude of this stress is:

$$\sigma_z = \frac{M_y x}{I_y} = \frac{(6.300\times10^6\text{ N-mm})(95\text{ mm}/2)}{1{,}435{,}806.017\text{ mm}^4} = 208.420\text{ MPa}$$

Moment M_z, which is a **torque**, creates a torsion shear stress in the *yz* plane at *K*. The magnitude of this shear stress is:

$$\tau_{yz} = \frac{M_z c}{J} = \frac{(3.360\times10^6\text{ N-mm})(95\text{ mm}/2)}{2{,}871{,}612.035\text{ mm}^4} = 55.579\text{ MPa}$$

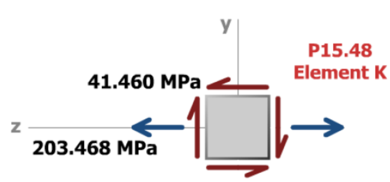
Summary of stresses at *K*:

$$\sigma_y = \boxed{0\text{ MPa}}$$

$$\sigma_z = -4.951\text{ MPa} + 208.420\text{ MPa} = 203.468\text{ MPa} \quad \boxed{203\text{ MPa (T)}}$$

$$\tau_{yz} = -14.118\text{ MPa} + 55.579\text{ MPa} = 41.460\text{ MPa} = \boxed{41.5\text{ MPa}}$$

Ans.



(b) Principal stress calculations:

$$\sigma_{p1,p2} = \frac{(203.468)\quad(0)}{2} \pm \sqrt{\left(\frac{(203.468)-(0)}{2}\right)^2 + (-41.460)^2}$$

$$= 101.734 \pm 109.858$$

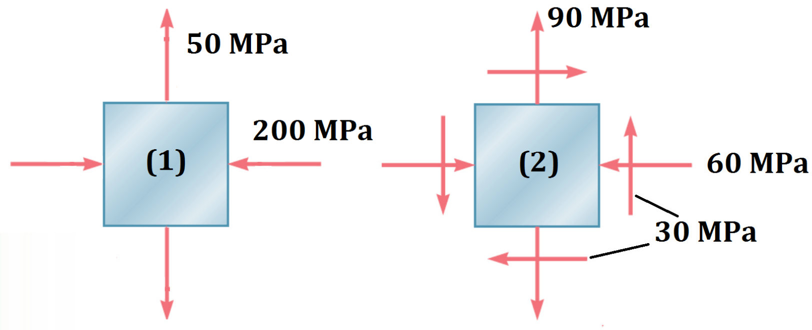
$$\boxed{\sigma_{p1} = 212\text{ MPa}} \quad \text{and} \quad \boxed{\sigma_{p2} = -8.12\text{ MPa}} \qquad \tau = 109.9\text{ MPa}$$

$$Tresca:\; FS = \frac{\tau_Y}{\tau_{max}} = \frac{250/2}{109.9} = 1.14$$

$$von\ Mises:\; \sigma_M^2 = \sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 = (212)^2 - (212)(-8.12) + (-8.12)^2$$

$$\sigma_M = 216.17\ MPa \qquad FS = \frac{250\ MPa}{216.17\ MPa} = 1.15$$

TBR 16: For a given brittle material under torsion the maximal shear stress is equal to 80 MPa. Moreover, the material fails under loading condition (1). Find factor of safety for this material under loading condition (2) (use Mohr's criterion).



$$\boldsymbol{Torsion}: \sigma_{1,2} = \pm 80\ MPa \rightarrow \frac{\sigma_1}{\sigma_{UT}} - \frac{\sigma_2}{\sigma_{UC}} = 1 \rightarrow \frac{80}{\sigma_{UT}} - \frac{-80}{\sigma_{UC}} = 1 \quad (1)$$

$$\boldsymbol{Loading\ 1}: \qquad \frac{50}{\sigma_{UT}} - \frac{-200}{\sigma_{UC}} = 1 \quad (2)$$

$$(1)\,and\,(2) \rightarrow \sigma_{UT} = 100\ MPa \qquad and \qquad \sigma_{UC} = 400\ MPa$$

$$\boldsymbol{Loading\ 2}:$$

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \frac{-60+90}{2} \pm \sqrt{\left(\frac{-60-90}{2}\right)^2 + 30^2}$$

$$\sigma_{1,2} = 95.78\ MPa, -65.78\ MPa$$

$$\frac{\sigma_1}{\sigma_{UT}} - \frac{\sigma_2}{\sigma_{UC}} = \frac{1}{FS} \rightarrow \frac{95.78}{100} - \frac{-65.78}{400} = \frac{1}{FS} \rightarrow FS = 0.89$$