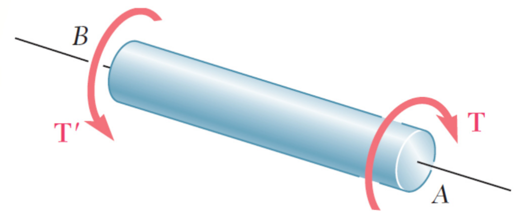
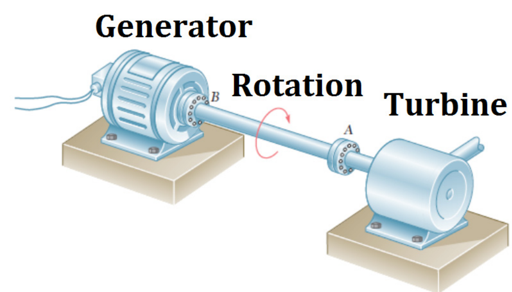
CHAPTER 3: TORSION

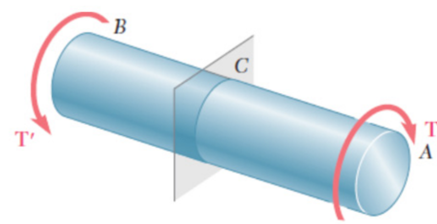


Introduction: In this chapter structural members and machine parts that are in *torsion* will be considered. More specifically, you will analyze the stresses and strains in members of circular cross section subjected to twisting moments, or *torques,* T and T'. Members in torsion are encountered in many engineering applications. The most common application is provided by *transmission shafts,* which are used to transmit power from one point to another. These shafts can be *solid* or *hollow*.

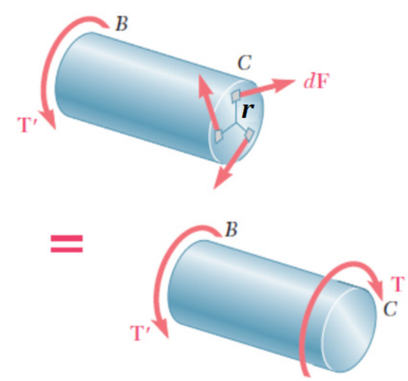


Analysis of Stress and Strain

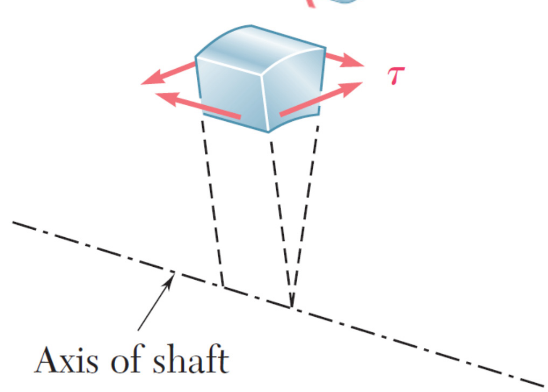
Now Consider a shaft *AB* subjected at *A* and *B* to equal and opposite torques **T** and **T'**, we pass a section perpendicular to the axis of the shaft through some arbitrary point *C* as shown. Based on the free-body diagram of the portion *BC* of the shaft and equilibrium we have:

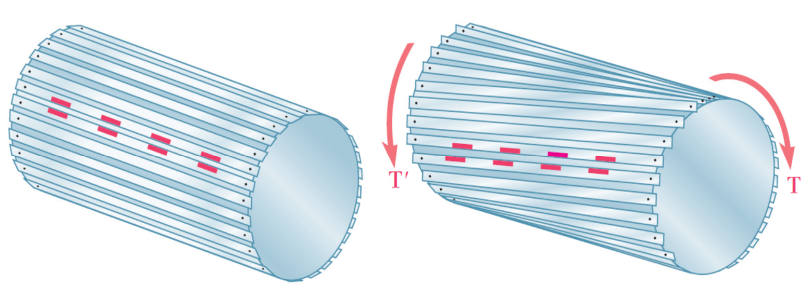


$$dT = r\,dF \rightarrow T = \int r\,dF = \int_A r\,\tau dA$$

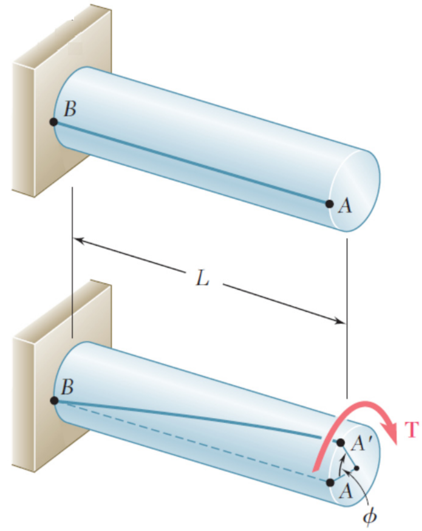


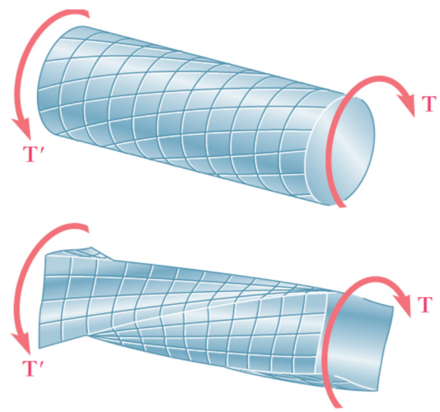
Also note that shear cannot take place in one plane only. Consider the very small element of shaft shown. We know that the torque applied to the shaft produces shearing stresses τ on the faces perpendicular to the axis of the shaft. But the conditions of equilibrium require the existence of equal stresses on the faces formed by the two planes containing the axis of the shaft. Such shearing stresses occur in torsion can be demonstrated by considering a "shaft" made of separate slats pinned at both ends to disks as shown below. If markings have been painted on two adjoining slats, it is observed that the slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft. While sliding will not actually take place in a shaft made of a homogeneous and cohesive material, the tendency for sliding will exist, showing that stresses occur on longitudinal planes as well as on planes perpendicular to the axis of the shaft.



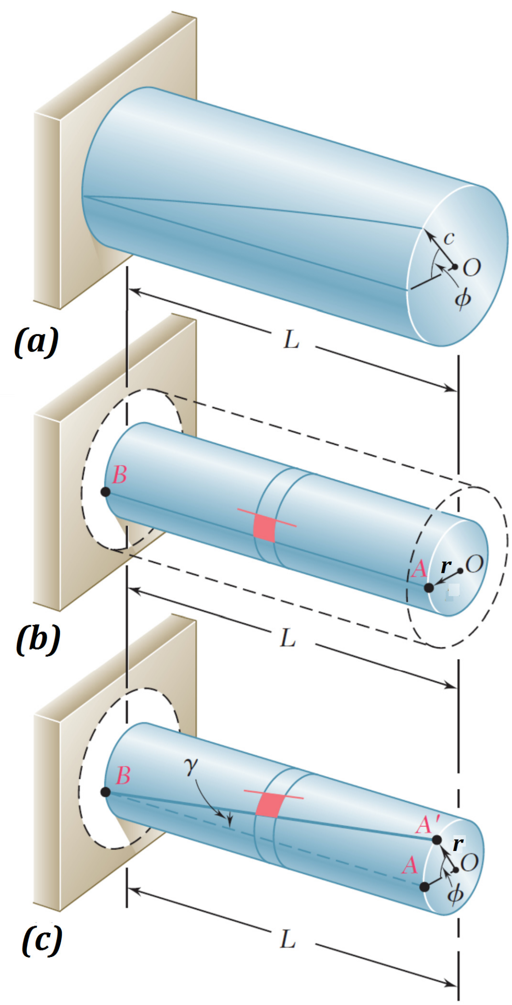


Now consider a circular shaft that is attached to a fixed support at one end. If a torque **T** is applied to the other end, the shaft will twist, with its free end rotating through an angle φ called *the angle of twist*. When a circular shaft is subjected to torsion, every cross section remains plane and undistorted. In other words, while the various cross sections along the shaft rotate through different amounts, each cross section rotates as a solid rigid slab (this is not the case for a shaft with square cross section as shown).





We will now determine the distribution of *shearing strains* in a circular shaft of length L and radius c that has been twisted through an angle φ. Detaching from the shaft a cylinder of radius r, we consider the small square element formed by two adjacent circles and two adjacent straight lines traced on the surface of the cylinder before any load is applied. As the shaft is subjected to a torsional load, the element deforms into a rhombus. Recall that the shearing strain γ in a given element is measured by the change in the angles formed by the sides of that element. Since the circles defining two of the sides of the element considered here remain unchanged, the shearing strain γ must be equal to the angle between lines *AB* and *A'B:*



$$r\varphi = L\gamma \rightarrow \gamma = \frac{r\varphi}{L} \rightarrow \gamma_{max} = \frac{c\varphi}{L} \rightarrow \gamma = \frac{r}{c}\gamma_{max}$$

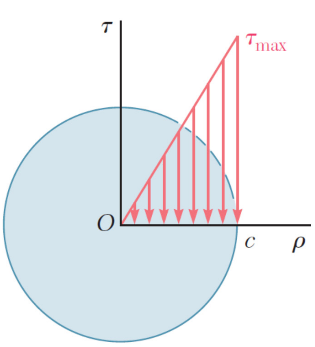
$$\tau = G\gamma \rightarrow \tau = \frac{r}{c}\tau_{max}$$

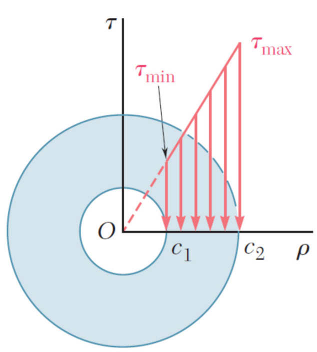
$$T = \int_A r\,\tau dA = \int_A r\,\frac{r}{c}\tau_{max}\,dA = \frac{\tau_{max}}{c}\int_A r^2 dA$$

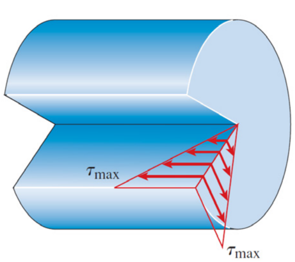
$$T = \frac{\tau_{max}}{c}J \rightarrow \tau_{max} = \frac{Tc}{J} \rightarrow \boldsymbol{\tau} = \frac{\boldsymbol{Tr}}{\boldsymbol{J}}$$

$$for\ solid\ shaft:\ J = \frac{\pi}{2}c^4, \qquad for\ hollow\ shaft:\ J = \frac{\pi}{2}(c_o^4 - c_i^4)$$

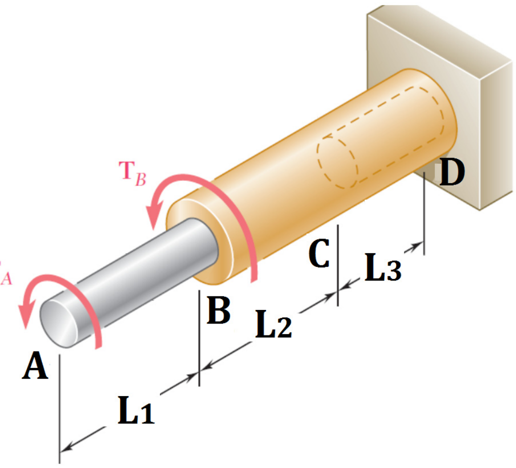
$$\tau = G\gamma \rightarrow \gamma = \frac{Tr}{GJ}\ and\ \varphi = \frac{L\gamma}{r} \rightarrow \boldsymbol{\varphi} = \frac{\boldsymbol{TL}}{\boldsymbol{JG}}$$







How can we measure G by a torsion test?



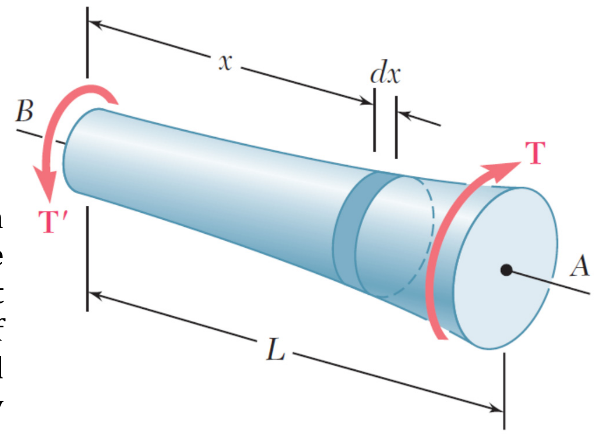
Shaft with intermediate torque

$$\varphi_A = \varphi_{A/B} + \varphi_{B/C} + \varphi_{C/D} + \varphi_D = \frac{TL}{JG})_{AB} + \frac{TL}{JG})_{BC} + \frac{TL}{JG})_{CD} + 0$$

$$\rightarrow \varphi_A = \sum_{i}^{n} \frac{T_i L_i}{J_i G_i}$$

Shaft with Continuously Varying Loads or Dimensions

$$d\varphi = \frac{T dx}{JG} \qquad \varphi = \int_0^L \frac{T(x)dx}{J(x)G(x)}$$

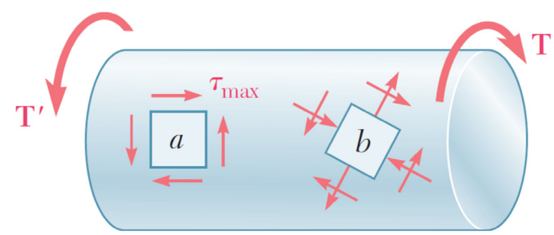


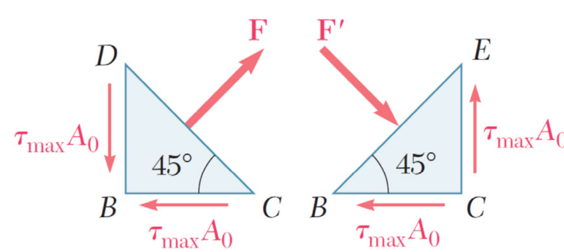
Normal Stress in Torsion

Up to this point, our analysis of stresses in a shaft has been limited to shearing stresses. This is due to the fact that the element we had selected was oriented in such a way that its faces were either parallel or perpendicular to the axis of the shaft. We know from earlier discussions that normal stresses, shearing stresses, or a combination of both may be found under the same loading condition, depending upon the orientation of the element that has been chosen. Consider the stresses and resulting forces on faces that are at 45° to the axis of the shaft (no shearing force acts along DC):

$$F = 2(\tau_{max} A_0) \cos 45° = \tau_{max} A_0 \sqrt{2}$$

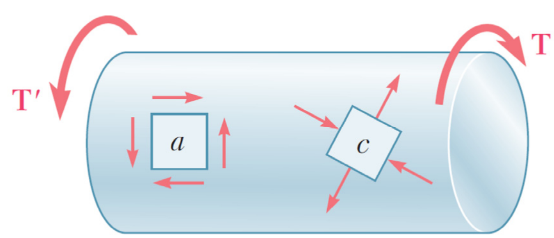
$$\sigma = \frac{F}{A} = \frac{\tau_{max} A_0 \sqrt{2}}{A_0 \sqrt{2}} = \tau_{max}$$



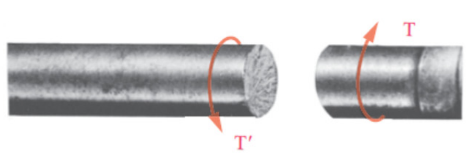


Failure model in Ductile and Brittle Materials

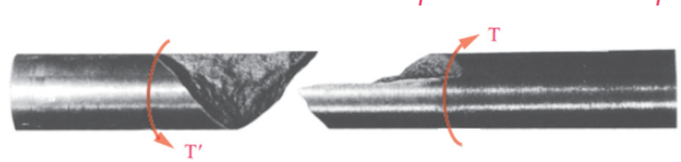
Ductile materials generally fail in shear. Therefore, when subjected to torsion, a specimen made of a ductile material breaks along a plane perpendicular to its longitudinal axis. On the other hand, brittle materials are weaker in tension than in shear. Thus, when subjected to torsion, a specimen made of a brittle material tends to break along surfaces that are perpendicular to the direction in which tension is maximum, i.e., along surfaces forming a 45^{o} angle with the longitudinal axis of the specimen.



$$\tau_{max} = \frac{Tc}{J} \qquad \sigma_{45°} = \pm \frac{Tc}{J}$$

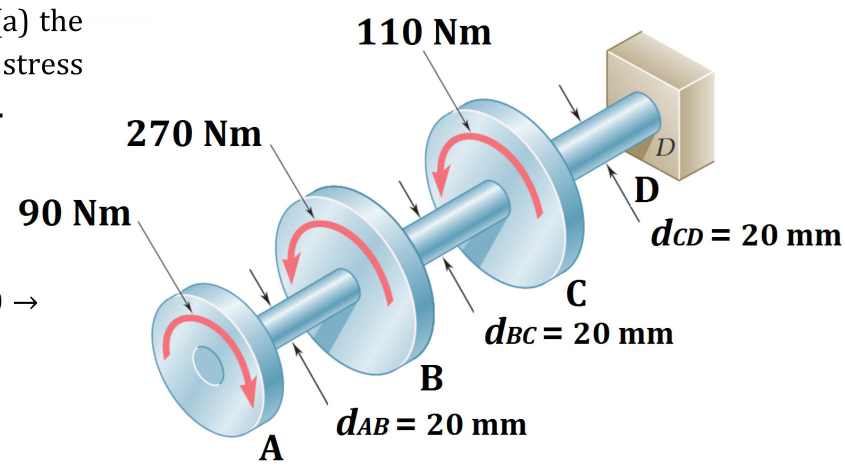


(a) Ductile Failure



(b) Brittle Failure

Example 1: Knowing that a 10 mm diameter hole is drilled through AD, determine (a) the shaft in which the maximum shearing stress occurs, (*b*) the magnitude of that stress.

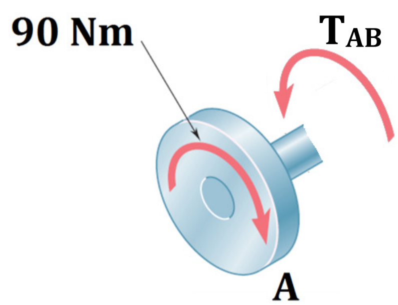


Part AB:

$$+\circlearrowleft \sum T = 0 \rightarrow T_{AB} - 90\ Nm = 0 \rightarrow$$

$$T_{AB} = 90\ Nm$$

$$\tau_{\max})_{AB} = \frac{T_{AB} r_{AB})_{max}}{J_{AB}} = \frac{(90\,000\ Nmm)(10\ mm)}{\frac{\pi}{2}(10^4 - 5^4)mm^4} = 61.1\ MPa$$

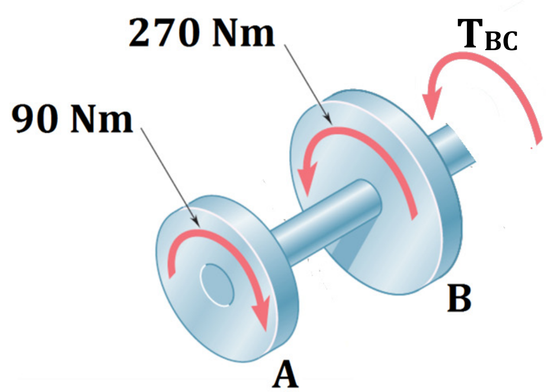


Part BC:

$$+\circlearrowleft \sum T = 0 \rightarrow T_{BC} + 270\ Nm - 90\ Nm = 0 \rightarrow$$

$$T_{BC} = -180\ Nm$$

$$\tau_{\max})_{BC} = \frac{T_{BC} r_{BC})_{max}}{J_{BC}} = \frac{(180\,000\ Nmm)(10\ mm)}{\frac{\pi}{2}(10^4 - 5^4)mm^4} = 122.2\ MPa$$

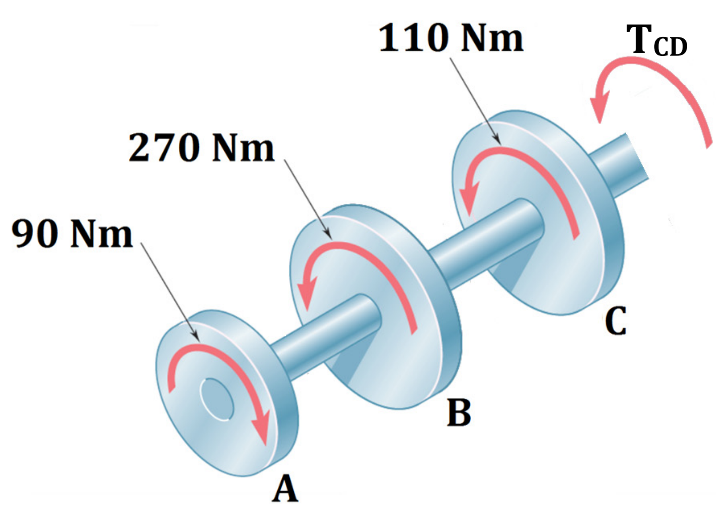


Part CD:

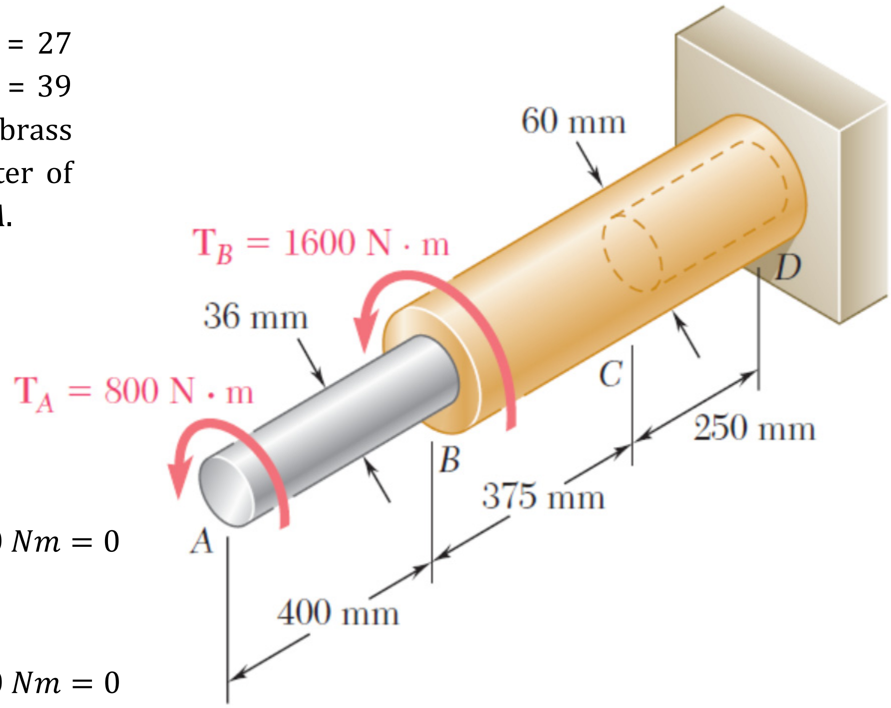
$$+\circlearrowleft \sum T = 0 \rightarrow T_{CD} - 90\ Nm + 270\ Nm + 110\ Nm = 0 \rightarrow$$

$$T_{CD} = -290\ Nm$$

$$\tau_{\max})_{CD} = \frac{T_{CD} r_{CD})_{max}}{J_{CD}} = \frac{(290\,000\ Nmm)(10\ mm)}{\frac{\pi}{2}(10^4 - 5^4)mm^4} = 196.9\ MPa \rightarrow \boldsymbol{Ans}$$



Example 2: The aluminium rod AB (G = 27 GPa) is bonded to the brass rod BD (G = 39 GPa). Knowing that portion CD of the brass rod is hollow and has an inner diameter of 40 mm, determine the angle of twist at A.



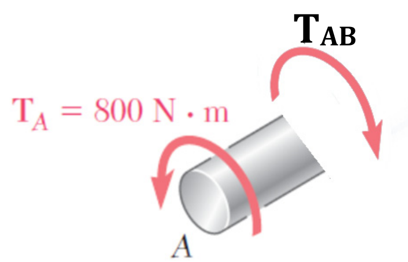
Statics:

$$+\curvearrowleft \sum T = 0 \rightarrow T_{AB} - 800\ Nm = 0 \rightarrow$$

$$T_{AB} = 800\ Nm$$

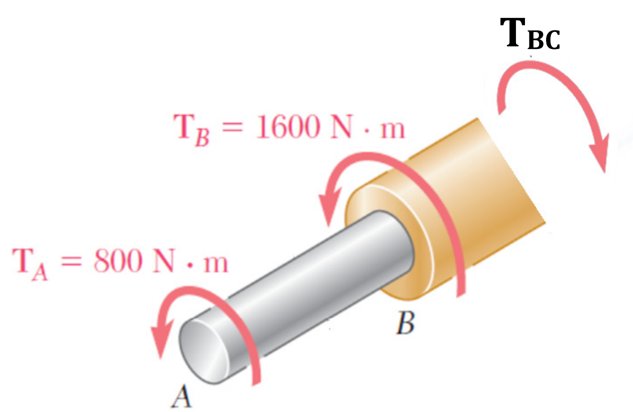
$$+\curvearrowleft \sum T = 0 \rightarrow T_{BC} - 800\ Nm - 1600\ Nm = 0$$
$$\rightarrow T_{BC} = 2400\ Nm$$

$$+\curvearrowleft \sum T = 0 \rightarrow T_{CD} - 800\ Nm - 1600\ Nm = 0$$
$$\rightarrow T_{CD} = 2400\ Nm$$

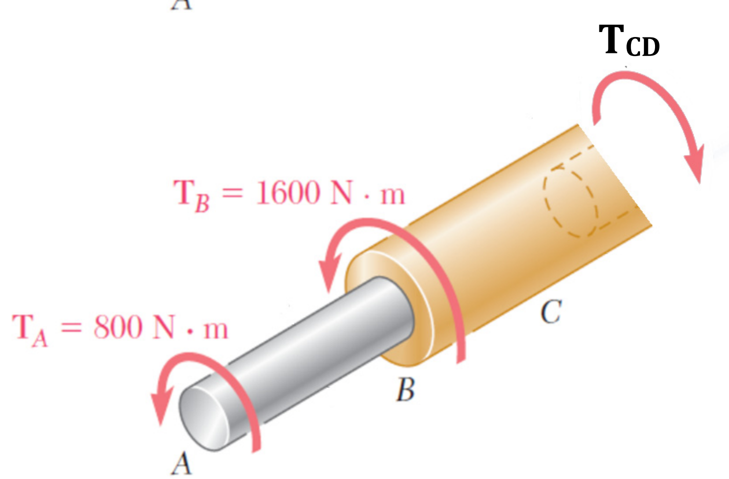


$$\varphi_A = \varphi_{A/B} + \varphi_{B/C} + \varphi_{C/D} + \varphi_D$$

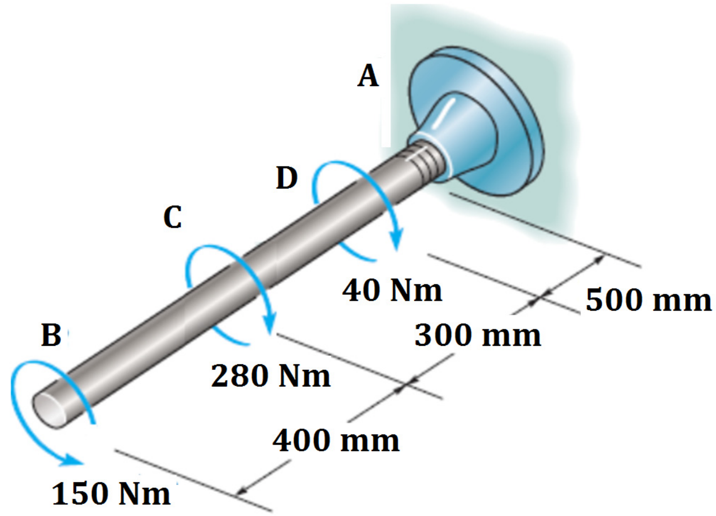
$$\varphi_A = \frac{TL}{JG})_{AB} + \frac{TL}{JG})_{BC} + \frac{TL}{JG})_{CD} + 0$$



$$\varphi_A = \frac{(800\,000\ Nmm)(400\ mm)}{\frac{\pi}{2}(18^4\ mm^4)(27\,000\ MPa)} + \frac{(2400\,000\ Nmm)(375\ mm)}{\frac{\pi}{2}(30^4\ mm^4)(39\,000\ MPa)} + \frac{(2400\,000\ Nmm)(250\ mm)}{\frac{\pi}{2}(30^4 - 20^4\ mm^4)(39\,000\ MPa)}$$
$$= 0.072^{rad} + 0.018^{rad} + 0.015^{rad} = 0.105^{rad}$$
$$= 6.02°$$



Example 3: The shaft (G = 80 GPa) has a diameter of 14 mm, determine the angle of twist at *B*.



$$+\curvearrowright \sum T = 0 \rightarrow T_{BC} - 150\ Nm = 0$$

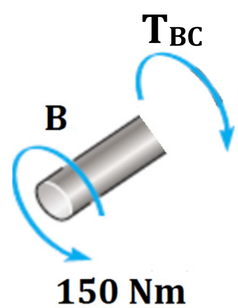
$$\rightarrow T_{BC} = 150\ Nm$$

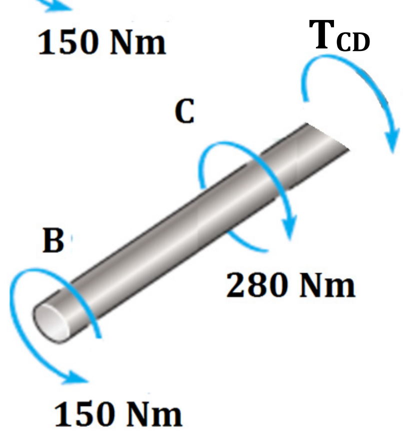
$$+\curvearrowright \sum T = 0 \rightarrow T_{CD} - 150\ Nm + 280\ Nm = 0$$

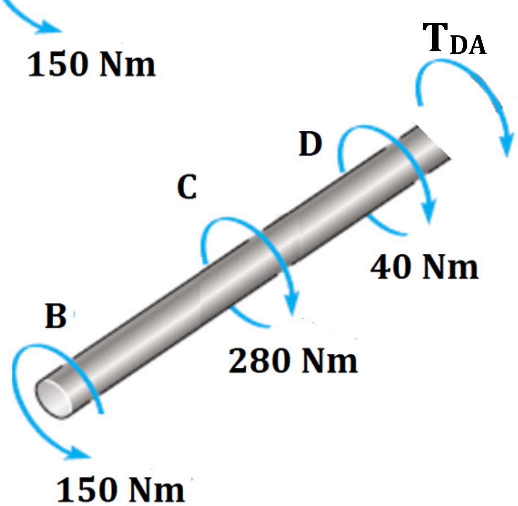
$$\rightarrow T_{CD} = -130\ Nm$$

$$+\curvearrowright \sum T = 0 \rightarrow T_{DA} - 150\ Nm + 280\ Nm + 40\ Nm = 0$$

$$\rightarrow T_{DA} = -170\ Nm$$







$$\varphi_B = \varphi_{B/C} + \varphi_{C/D} + \varphi_{D/A} + \varphi_A$$

$$\varphi_B = \frac{TL}{JG})_{BC} + \frac{TL}{JG})_{CD} + \frac{TL}{JG})_{DA} + 0$$

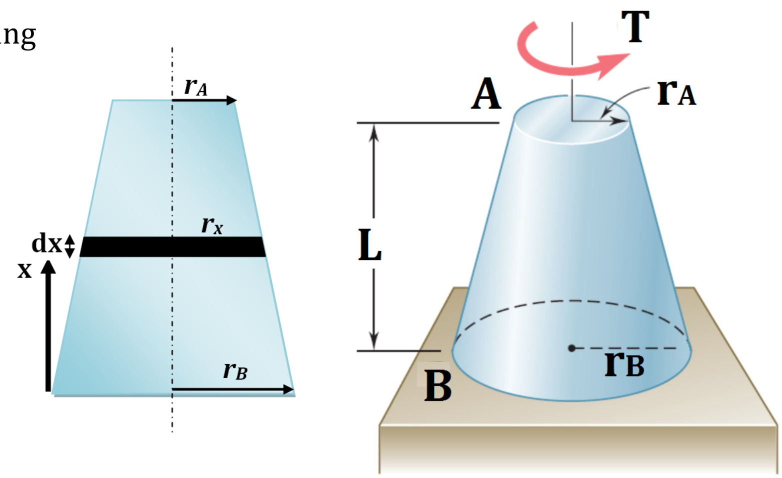
$$\varphi_B = \frac{(150\,000\ Nmm)(400\ mm)}{\frac{\pi}{2}(7^4 mm^4)(80\,000\ MPa)} + \frac{(-130\,000\ Nmm)(300\ mm)}{\frac{\pi}{2}(7^4 mm^4)(80\,000\ MPa)} + \frac{(-170\,000\ Nmm)(500\ mm)}{\frac{\pi}{2}(7^4 mm^4)(80\,000\ MPa)}$$

$$= 0.2^{rad} - 0.13^{rad} - 0.28^{rad} = -0.21^{rad}$$

$$= -12.1^\circ = 12.1^\circ \curvearrowright$$

Example 4: For the given shaft and loading show that:

$$\varphi_A = \frac{2TL\,(r_B^2 + r_B r_A + r_A^2)}{3\pi G\, r_A^3 r_B^3}$$



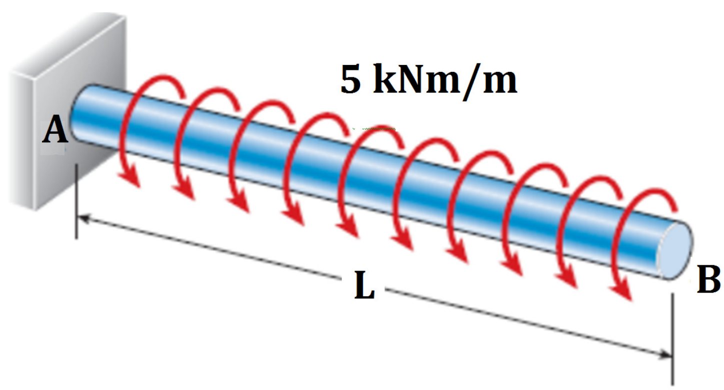
$$d\varphi = \frac{T dx}{J_x G} \rightarrow \varphi_A = \int_0^L \frac{T dx}{J_x G} = \frac{T}{G}\int_0^L \frac{dx}{J_x}$$

$$J_x = \frac{\pi}{2} r_x^4, \quad r_x = \frac{r_A - r_B}{L} x + r_B$$

$$\rightarrow \varphi_A = \frac{T}{G}\int_0^L \frac{dx}{\frac{\pi}{2}\left(\frac{r_A - r_B}{L} x + r_B\right)^4}$$

$$\varphi_A = \frac{2TL\,(r_B^2 + r_B r_A + r_A^2)}{3\pi G\, r_A^3 r_B^3}$$

Example 5: For the shaft and loading shown (G= 75 GPa, d = 80 mm, L = 800 mm) determine the angle of twist at B.



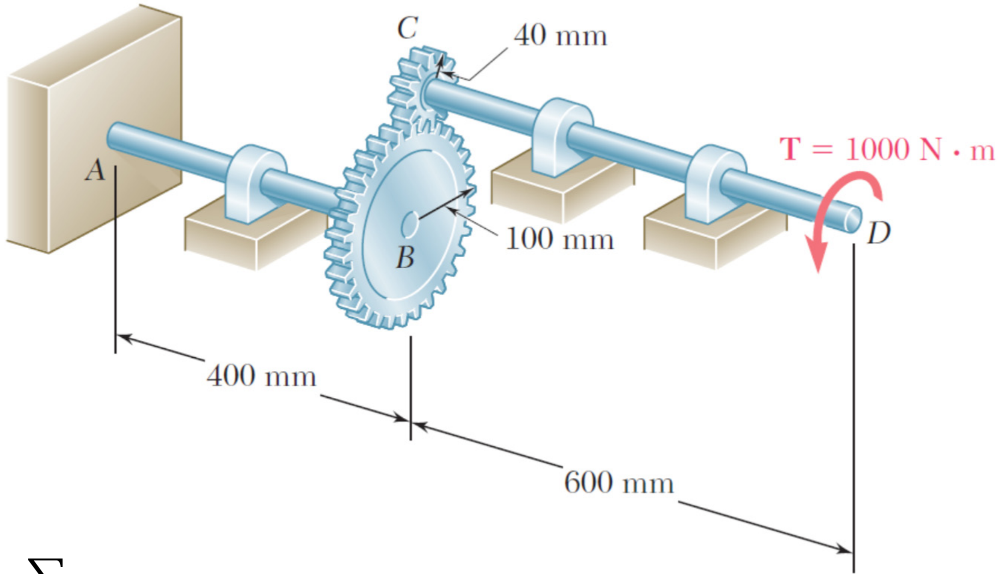
$$d\varphi = \frac{T_x dx}{JG} \rightarrow \varphi_B = \int_0^L \frac{T_x dx}{JG}$$

$$= \frac{1}{JG}\int_0^L T_x dx = \frac{1}{\left(\frac{\pi}{2} 40^4 mm^4\right)(75\,000\ MPa)}\int_0^L (5000\ Nmm/mm \times x)\, dx$$

$$= \frac{5000\ \ Nmm/mm}{\left(\frac{\pi}{2} 40^4 mm^4\right)(75\,000\ MPa)}\int_0^L (x)\, dx = \frac{(5000\ Nmm/mm)(800\ mm)^2/2}{\left(\frac{\pi}{2} 40^4 mm^4\right)(75\,000\ MPa)}$$

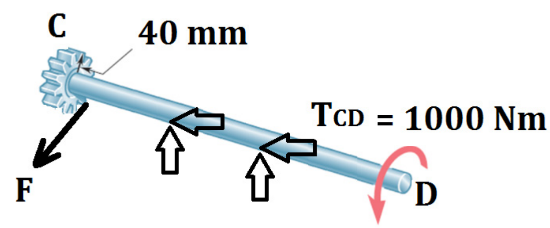
$$= 0.0053^{rad} = 0.3°$$

Example 6: The design of the gear-and-shaft system shown requires that steel shafts of the same diameter be used for both *AB* and *CD*. It is further required that $\tau_{max} \leq 60$ MPa and that the angle φ_D through which end *D* of shaft *CD* rotates not exceed 1.5°. Knowing that *G* = 77 GPa, determine the required diameter of the shafts.



$$\sum T_C = 0 \rightarrow T_{CD} - r_C F = 0 \qquad \sum T_B = 0 \rightarrow T_{AB} - r_B F = 0$$

$$\rightarrow T_{AB} = \frac{r_B}{r_C} T_{CD} = \frac{100\ mm}{40\ mm} \times 1000\ Nm = 2500\ Nm$$



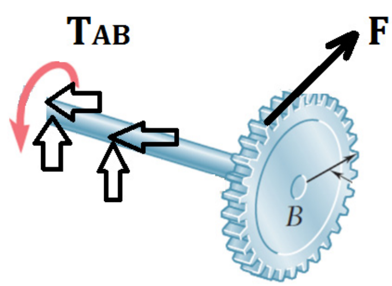
Shear stress remaining smaller than 60 MPa:

Maximum shear stress occurs in shaft AB as $T_{AB} > T_{CD}$

$$\tau_{max} = \frac{T_{AB} r}{J_{AB}} \leq 60\ MPa \rightarrow$$

$$\frac{(2500\,000\ Nmm)\, r}{\frac{\pi}{2} r^4} \leq 60\ MPa \rightarrow r \geq 29.82\ mm \ \rightarrow d \geq 59.64\ mm$$

Angle of twist at D remaining smaller than 1.5°:



$$\varphi_D = \varphi_{D/C} + \varphi_C \rightarrow \frac{1.5\pi}{180} = \frac{T_{CD} L_{CD}}{J_{CD} G_{CD}} + \varphi_C$$

$$\frac{1.5\pi}{180} = \frac{T_{CD} L_{CD}}{J_{CD} G_{CD}} + \varphi_C \rightarrow \frac{1.5\pi}{180} = \frac{(1000\,000\ Nmm)(600\ mm)}{\frac{\pi}{2} r^4 \times 77\,000\ MPa} + \varphi_C$$

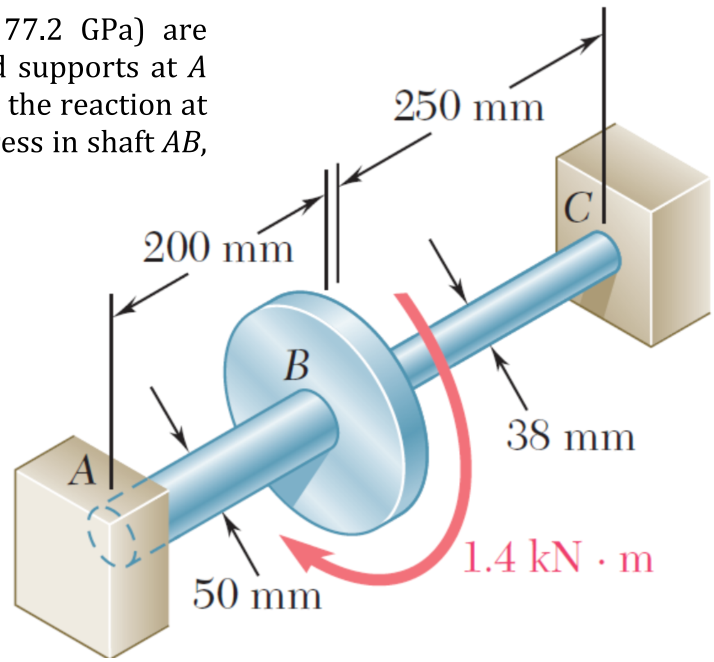
$$0.02618 = \frac{4961}{r^4} + \varphi_C\ (two\ unkonws: r\ and\ \varphi_C)$$

$$r_C \varphi_C = r_B \varphi_B \rightarrow \varphi_C = \frac{r_B}{r_C} \varphi_B \ \ and\ also\ \varphi_B = \varphi_{B/A} + \varphi_A = \frac{T_{AB} L_{AB}}{J_{AB} G_{AB}}$$

$$\rightarrow \varphi_B = \frac{(2500\,000\ Nmm)(400\ mm)}{\frac{\pi}{2} r^4 \times 77\,000\ MPa} \rightarrow \varphi_B = \frac{8268}{r^4} \rightarrow \varphi_C = \frac{100\ mm}{40\ mm} \times \frac{8268}{r^4} = \frac{20670}{r^4}$$

$$0.02618 = \frac{4961}{r^4} + \varphi_C \rightarrow 0.02618 = \frac{4961}{r^4} + \frac{20670}{r^4} \rightarrow r = 31.45\ mm \rightarrow d \geq 62.9\ mm \quad (\boldsymbol{Ans})$$

Example 7: Two solid steel shafts (G = 77.2 GPa) are connected to a coupling disk B and to fixed supports at A and C. For the loading shown, determine (a) the reaction at each support, (b) the maximum shearing stress in shaft AB, (c) the maximum shearing stress in shaft BC.



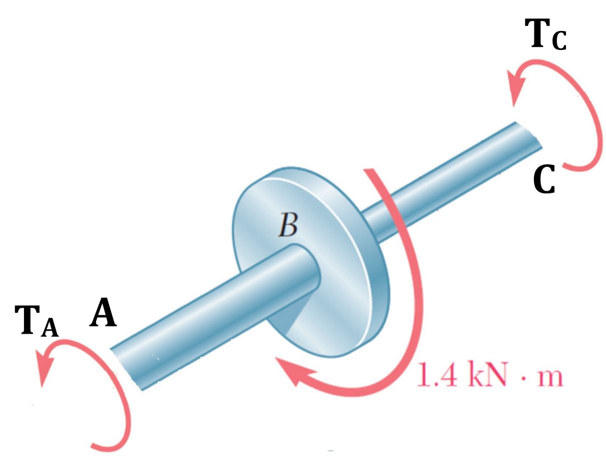
$$\sum T = 0 \rightarrow T_A + T_C = 1.4\ kNm$$

The system is statically indeterminate.

Compatibility equation:

$$\varphi_C = 0 \rightarrow \varphi_{C/B} + \varphi_{B/A} + \varphi_A = 0$$

$$\frac{T_{BC}L_{BC}}{J_{BC}G} + \frac{T_{AB}L_{AB}}{J_{AB}G} + 0 = 0$$



$$\sum T = 0 \rightarrow T_{BC} = T_C$$

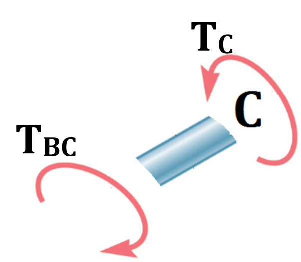
$$\sum T = 0 \rightarrow T_{AB} = T_C - 1.4\ kNm$$

$$\frac{T_C \times 250\ mm}{\frac{\pi}{2}19^4} + \frac{(T_C - 1\,400\,000\ Nmm) \times 200\ mm}{\frac{\pi}{2}25^4} = 0$$

$$\rightarrow T_C = 294938\ Nmm = 295\ Nm$$

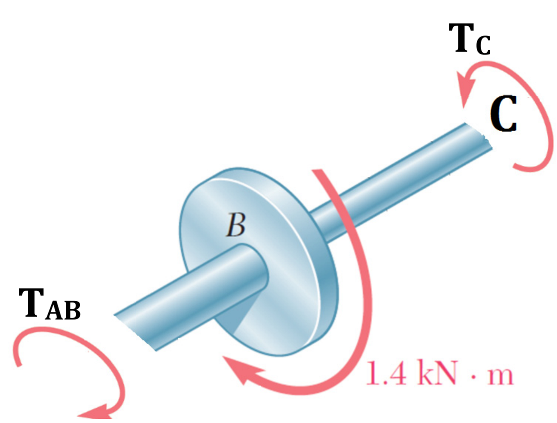
$$\rightarrow T_A = 1400\ Nm - 295\ Nm = 1105\ Nm$$

$$\rightarrow T_{BC} = T_C = 295\ Nm$$



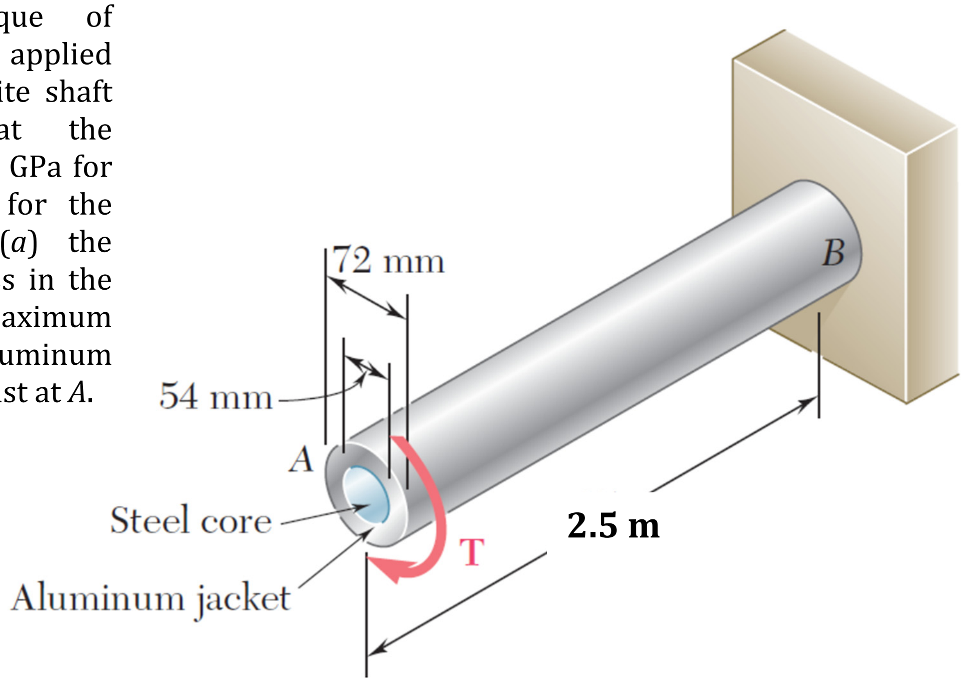
$$\rightarrow T_{AB} = T_C - 1.4\ kNm = 295\ Nm - 1400\ Nm$$
$$= -1105\ Nm$$

$$\tau_{AB\,max} = \frac{T_{AB}r_{AB}}{J_{AB}} = \frac{(1105\,000\ Nmm)(25\ mm)}{\frac{\pi}{2}25^4 mm^4} = 45\ MPa$$



$$\tau_{BC\,max} = \frac{T_{BC}r_{BC}}{J_{BC}} = \frac{(295\,000\ Nmm)(19\ mm)}{\frac{\pi}{2}19^4 mm^4} = 27.4\ MPa$$

Example 8: A torque of magnitude T = 4 kNm is applied at end A of the composite shaft shown. Knowing that the modulus of rigidity is 77 GPa for the steel and 27 GPa for the aluminum, determine (a) the maximum shearing stress in the steel core, (b) the maximum shearing stress in the aluminum jacket, (c) the angle of twist at A.



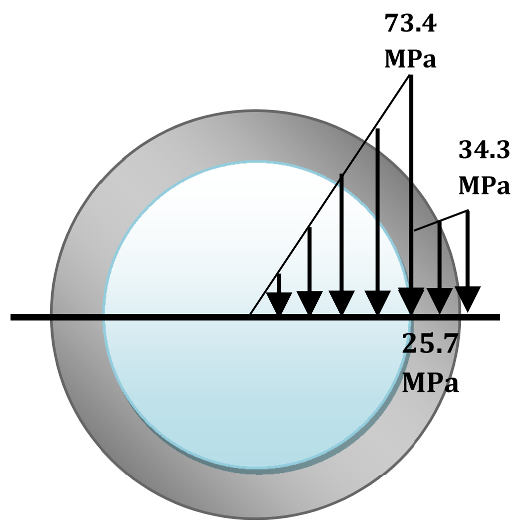
$$Equilibrium: \quad T = T_A + T_S = 4\ kNm \ \ (1) \rightarrow statically\ indeterminate$$

Compatibility equation:

$$\varphi_A = \varphi_S \rightarrow \frac{T_A L}{J_A G_A} = \frac{T_S L}{J_S G_S} \rightarrow \frac{T_A}{\frac{\pi}{2}(36^4 - 27^4)\ mm^4 \times 27\ 000\ MPa} = \frac{T_S}{\frac{\pi}{2}(27^4)\ mm^4 \times 77\ 000\ MPa} \rightarrow$$

$$T_S = 1.32\ T_A \quad (2)$$

$$\xrightarrow{(1) and\ (2)} T_A = 1.72\ kNm \quad and \quad T_S = 2.27\ kNm$$



Maximum shear stress in the steel core:

$$\tau_S)_{max} = \frac{T_S r_{max}}{J_S} = \frac{(2.27 \times 10^6\ Nmm)(27\ mm)}{\frac{\pi}{2}(27^4)\ mm^4} = 73.42\ MPa$$

Maximum shear stress in the Aluminium jacket:

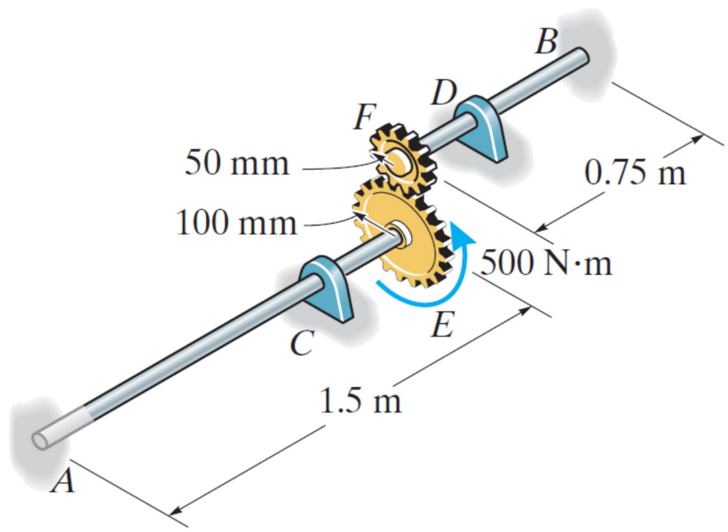
$$\tau_A)_{max} = \frac{T_A r_{max}}{J_A} = \frac{(1.72 \times 10^6\ Nmm)(36\ mm)}{\frac{\pi}{2}(36^4 - 27^4)\ mm^4} = 34.3\ MPa$$

The angle of twist at A:

$$\varphi_A = \frac{T_A L}{J_A G_A} = \frac{(1.72 \times 10^6\ Nmm)(25\ 00\ mm)}{\frac{\pi}{2}(36^4 - 27^4)\ mm^4 \times 27\ 000\ MPa} = 0.088^{rad} = 5.05°$$

TBR 1: shafts are made of A-36 steel (G = 75 GPa). Each has a diameter of 25 mm and they are connected using the gears fixed to their ends. Their other ends are attached to fixed supports at *A* and *B*. They are also supported by journal bearings at *C* and *D*, which allow free rotation of the shafts along their axes. If a torque of 500 Nm is applied to the gear at *E* as shown, determine the reactions at *A* and *B* as well as the angle of twist at E (1390).

Answer: T_B = 222.22 Nm, T_A=55.6 Nm, φ_E=1.66°

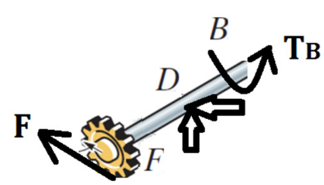


From Statics (equilibrium) of torques:

$$AE: \sum T = 0 \rightarrow -T_A + 500 - r_E F = 0 \quad (1)$$

$$BF: \sum T = 0 \rightarrow T_B - r_F F = 0 \quad (2)$$

We have two equations three unknowns (T_A, T_B, and F), so we need a compatibility equation:

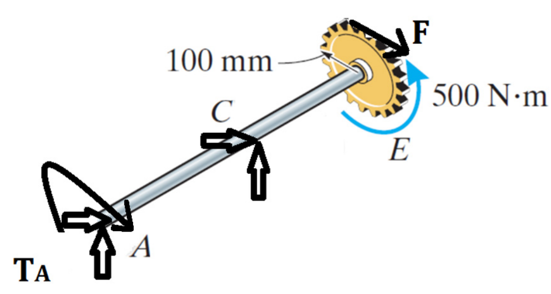


$$r_E \varphi_E = r_F \varphi_F \rightarrow r_E(\varphi_{E/A} + \varphi_A) = r_F((\varphi_{F/B} + \varphi_B) \rightarrow$$

$$r_E \left(\frac{T_{EA}L_{EA}}{J_{EA}G_{EA}} + 0\right) = r_F \left(\frac{T_{FB}L_{FB}}{J_{FB}G_{FB}} + 0\right)$$

$$\rightarrow 100\ mm \left(\frac{T_A \times 1500\ mm}{JG}\right) = 50\ mm \left(\frac{T_B \times 750\ mm}{JG}\right) \rightarrow$$

$$\rightarrow T_B = 4\,T_A \quad (3)$$



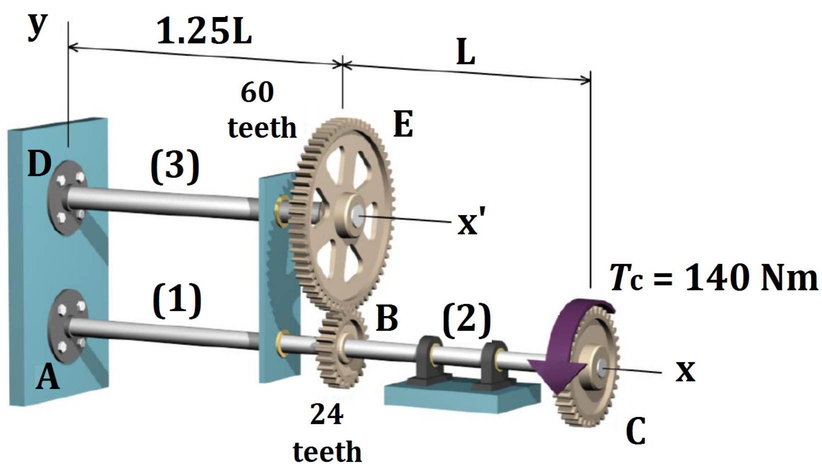
$$\xrightarrow{(1),(2),\ and\ (3)} T_A = 55.6\ Nm, T_B = 222.22\ Nm$$

$$\varphi_E = \varphi_{E/A} + \varphi_A = \varphi_{E/A} = \frac{T_{EA}L_{EA}}{J_{EA}G_{EA}} = \frac{T_A L_{EA}}{J_{EA}G_{EA}}$$

$$= \frac{(55600\ Nmm)(1500\ mm)}{\frac{\pi}{2}(12.5\ mmm)^4(75\ 000\ MPa)} = 0.029\ rad$$

$$= 1.66°$$

TBR 2: shafts (1) and (2) have a diameter of 20 mm and shaft (3) has a diameter of 25 mm. The supports allow free rotation of the shafts along their axes. Determine the maximal shear stress in shaft (1). Also, find rotation of gears C and E. Assume that G = 80 GPa and L = 400 mm (1391).



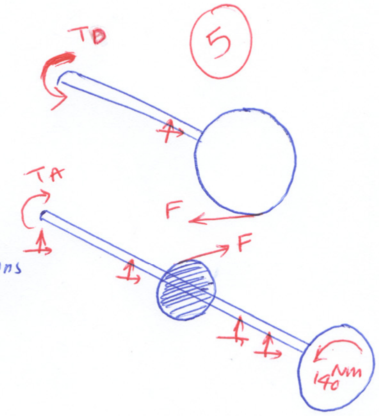
$\Sigma T = 0.$

ABC :

$140 - 24F - T_A = 0.$ (1) 2

DE :

$60F - T_D = 0.$ (2) 2



Two equations three unknowns

compatibility !

$24\,\varphi_B = 60\,\varphi_E$ 5

$\because \varphi_E = \varphi_{E/D} + \cancel{\varphi_D}^{\,0} = \varphi_{E/D} = \dfrac{T_{ED}L_{ED}}{J_{ED}G_{ED}} = \dfrac{T_D L_{ED}}{J_{ED}G_{ED}}$ 3

$\varphi_B = \varphi_{B/A} + \cancel{\varphi_A}^{\,0} = \varphi_{B/A} = \dfrac{T_{AB}L_{AB}}{J_{AB}G_{AB}} = \dfrac{T_A L_{AB}}{J_{AB}G_{AB}}$ 3

$\rightarrow 24\dfrac{T_A \cancel{L_{AB}}}{J_{AB}\cancel{G_{AB}}} = 60\dfrac{T_D \cancel{L_{ED}}}{J_{ED}\cancel{G_{ED}}} \rightarrow \dfrac{24\,T_A}{\frac{\pi}{2}\left(\frac{20}{2}\right)^4} = \dfrac{60\,T_D}{\frac{\pi}{2}\left(\frac{25}{2}\right)^4}$

$\rightarrow T_D = 0.976\,T_A$ (3) 1 $\quad$ (1), (2), (3) $\rightarrow$ $T_A = 100.69$ Nm 1, $T_D = 98.27$ Nm

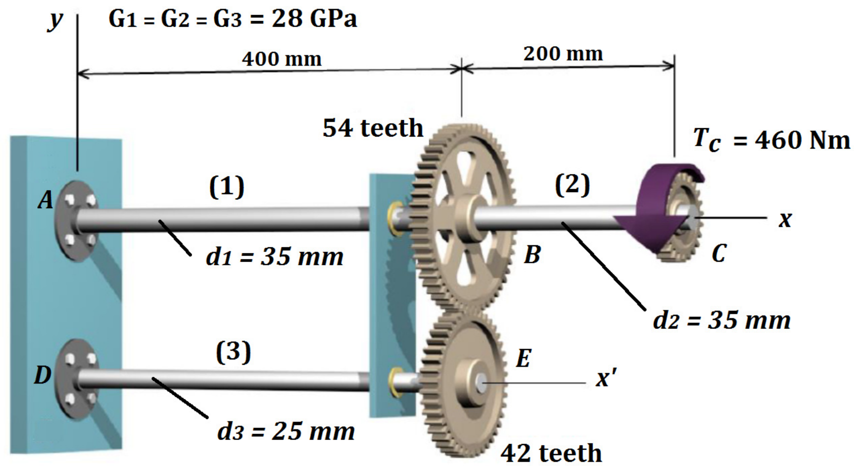
$\tau_1 = \dfrac{T_{AB}\,r_{AB}}{J_{AB}} = \dfrac{(100.69\times10^3\ \text{Nmm})(20\text{mm}/2)}{\frac{\pi}{2}\left(20/2\ \text{mm}\right)^4} = 64.1\ \text{MPa}$ 3

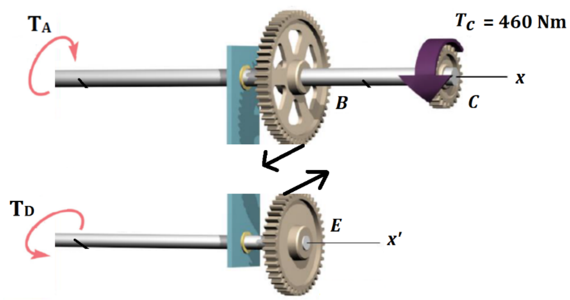
$\varphi_C = \varphi_{C/B} + \varphi_{B/A} + \cancel{\varphi_A}^{\,0} = \dfrac{T_C L_{BC}}{J_{BC}G_{BC}} + \dfrac{T_A L_{AB}}{J_{AB}G_{AB}} =$

$\dfrac{(140000\,\text{Nmm})(400\,\text{mm})}{\frac{\pi}{2}(10\text{mm})^4(80000\,\text{MPa})} + \dfrac{(100.69\times10^3\,\text{Nmm})(1.25\times400\,\text{mm})}{\frac{\pi}{2}(10\,\text{mm})^4\ (80000\,\text{MPa})} = 0.0846\ \text{rad} = 4.84^\circ$ 3

$\varphi_E = \dfrac{T_D L_{ED}}{J_{ED}G_{ED}} = \dfrac{(98.27\times10^3\ \text{Nmm})(1.5\times400\ \text{mm})}{\frac{\pi}{2}\left(\frac{25}{2}\right)^4\text{mm}^4\times 80\times10^3\ \text{MPa}} = 0.0192\ \text{rad} = 1.101^\circ$ 2

TBR 3: Find the maximal shear stress in shafts (1) and (3) as well as rotation of gears C and E (1392).





$$T_C - F \times r_B - T_A = 0 \;\; (Equilibrium\ of\ shaft\ ABC), T_D - F \times r_E = 0 \; (Equilibrium\ of\ shaft\ DE)$$

$$\rightarrow \frac{T_C - T_A}{T_D} = \frac{r_B}{r_E} = \frac{54}{42} = 1.286 \rightarrow \frac{460 - T_A}{T_D} = 1.286 \rightarrow 1.286 T_D + T_A = 460\ Nm \quad (1)$$

$\rightarrow$ *Statically indeterminate*: 2 *unknowns* (T_A *and* T_D): Compatibility Equations:

$$r_B \varphi_B = r_E \varphi_E \rightarrow r_B(\varphi_{B/A} + \varphi_A) = r_E(\varphi_{E/D} + \varphi_D) \xrightarrow{\varphi_A = \varphi_D = 0} r_B(\varphi_{B/A}) = r_E(\varphi_{E/D})$$

$$r_B\left(\frac{T_{AB} L_{AB}}{J_{AB} G_{AB}}\right) = r_E\left(\frac{T_{DE} L_{DE}}{J_{DE} G_{DE}}\right) \rightarrow r_B\left(\frac{T_A L_{AB}}{J_{AB} G_{AB}}\right) = r_E\left(\frac{T_D L_{DE}}{J_{DE} G_{DE}}\right) \xrightarrow{L_{AB} = L_{DE}\ and\ G_{AB} = G_{DE}} r_B\left(\frac{T_A}{J_{AB}}\right) = r_E\left(\frac{T_D}{J_{DE}}\right)$$

$$54\left(\frac{T_A}{\pi\left(\frac{35}{2}\right)^4}\right) = 42\left(\frac{T_D}{\pi\left(\frac{25}{2}\right)^4}\right) \rightarrow T_A = 2.988\ T_D \;\; (2) \xrightarrow{(1) and\ (2)} T_D = 107.6\ Nm,\ T_A = 321.6\ Nm$$

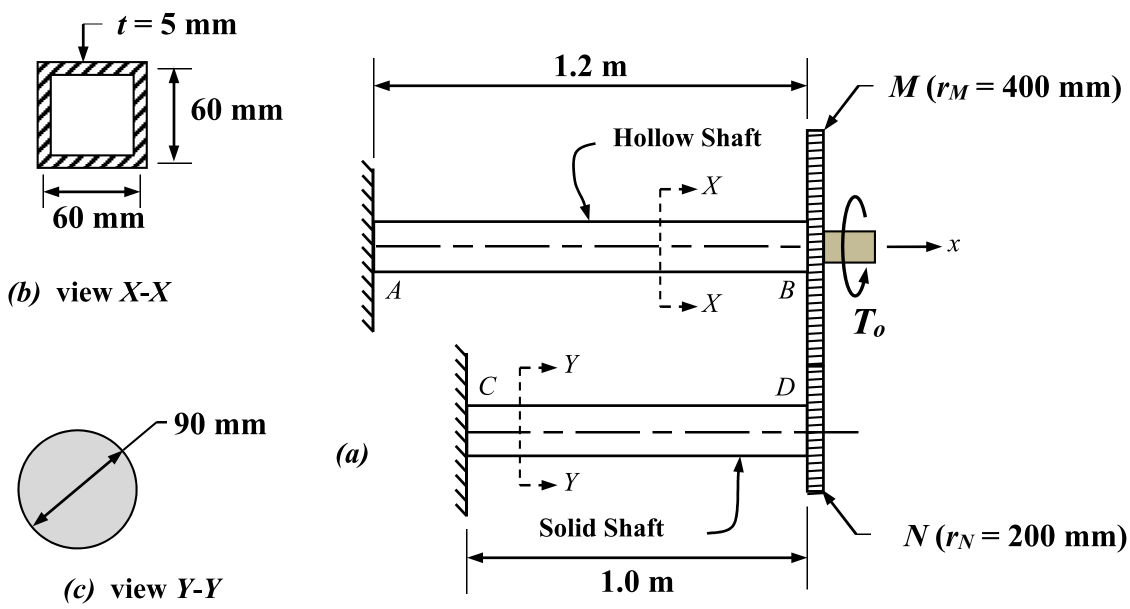
$$\tau_1 = \frac{321.6 \times 10^3 Nmm \times \frac{35}{2} mm}{\frac{\pi}{2}\left(\frac{35}{2} mm\right)^4} = 38.2\ MPa, \tau_2 = \frac{107.6 \times 10^3 Nmm \times \frac{25}{2} mm}{\frac{\pi}{2}\left(\frac{25}{2} mm\right)^4} = 35.1\ MPa$$

$$\varphi_E = \varphi_{E/D} + \varphi_D = \varphi_{E/D} = \frac{T_{DE} L_{DE}}{J_{DE} G_{DE}} = \frac{T_D L_{DE}}{J_{DE} G_{DE}} = \frac{(107.6 \times 10^3\ Nmm)\ (400\ mm)}{\frac{\pi}{2}\left(\frac{25}{2}\right)^4 \times 28000\ MPa} = 0.04\ rad = 2.3°$$

$$\varphi_C = \varphi_{C/B} + \varphi_{B/A} + \varphi_A = \varphi_{C/B} + \varphi_{B/A} = \frac{T_{BC} L_{BC}}{J_{BC} G_{BC}} + \frac{T_{AB} L_{AB}}{J_{AB} G_{AB}}$$

$$= \frac{(460\,000\ Nmm)(200\ mm)}{\frac{\pi}{2}\left(\frac{35}{2}\right)^4 \times 28000\ MPa} + \frac{(321.6 \times 10^3\ Nmm)(400\ mm)}{\frac{\pi}{2}\left(\frac{35}{2}\right)^4 \times 28000\ MPa} = 0.0535\ rad = 3.06°$$

TBR 4: For the gear system shown find maximal T_0 so that maximal shear stress in the thin-walled shaft AB remains smaller than 40 MPa. Based on the calculated T_0 determine rotations of N and M as well as maximal stress in shoft CD (G= 60 GPa) (1393).



$T_0 - 400F - T_A = 0$ 2

$200F - T_C = 0$ 2

$\rightarrow 2T_C + T_A = T_0 \;\; (1) \quad \textit{Statically Indeterminate}$ 1

$\textit{Compatibility}: 400 \times \varphi_M = 200 \times \varphi_N \rightarrow \varphi_N = 2\varphi_M$ 2 $\quad \rightarrow \frac{TL}{JG})_{CD} = 2 \times \frac{TL}{4\bar{A}^2 G} \oint \frac{ds}{t})_{AB}$

$$\rightarrow \frac{T_C(1000\ mm)}{\frac{\pi}{2}(45\ mm)^4 \times 60\ 000\ MPa} = 2 \times \frac{T_A(1200\ mm)}{4 \times 3600^2 mm^2 \times 60\ 000\ MPa}\left(\frac{4 \times 60\ mm}{5\ mm}\right)$$ 8

$0.000155\ T_C = 0.002222\ T_A \rightarrow T_C = 14.31\ T_A \quad (2)$ 1

$(1)\ and\ (2) \quad \rightarrow T_A = 0.03375\ T_0 \qquad T_C = 0.483\ T_0$ 2

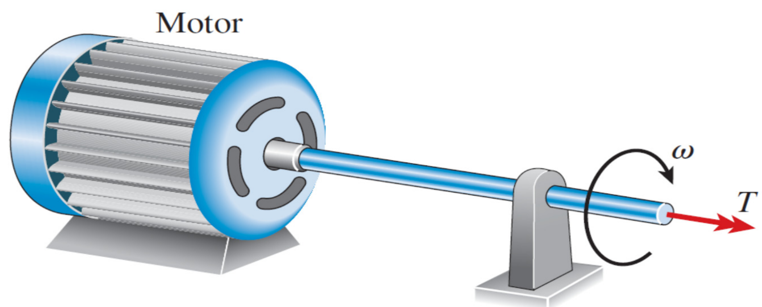
$$\tau_{AB} = \frac{T_A}{2\bar{A}t} \rightarrow 40\ MPa = \frac{0.03375 T_0}{2\ (3600\ mm^2)(5\ mm)} \rightarrow T_0 = 42\ 663\ 978\ Nmm = 42.6\ kNm$$ 4

$$\tau_{CD} = \frac{T_C r_{CD}}{J_{CD}} = \frac{0.483 \times 42\ 663\ 978\ Nmm \times 45\ mm}{\frac{\pi}{2}(45\ mm)^4} = 144\ MPa$$ 2 $$\varphi_N = \frac{T_C\ (1000\ mm)}{\frac{\pi}{2}(45\ mm)^4 \times 60\ 000\ MPa}$$

$$= 0.0533\ rad = 3.05°, \;\; \varphi_M = \frac{\varphi_N}{2} = 1.52°$$ 1

Design of Transmission Shafts

The principal specifications to be met in the design of a transmission shaft are the *power* to be transmitted and the *speed of rotation* of the shaft. The role of the designer is to select the material and the dimensions of the cross section of the shaft, so that the maximum shearing stress allowable in the material will not be exceeded when the shaft is transmitting the required power at the specified speed.



To determine the torque exerted on the shaft, we recall from elementary dynamics that the power P associated with the rotation of a rigid body subjected to a torque T is:

$$P = T\omega$$

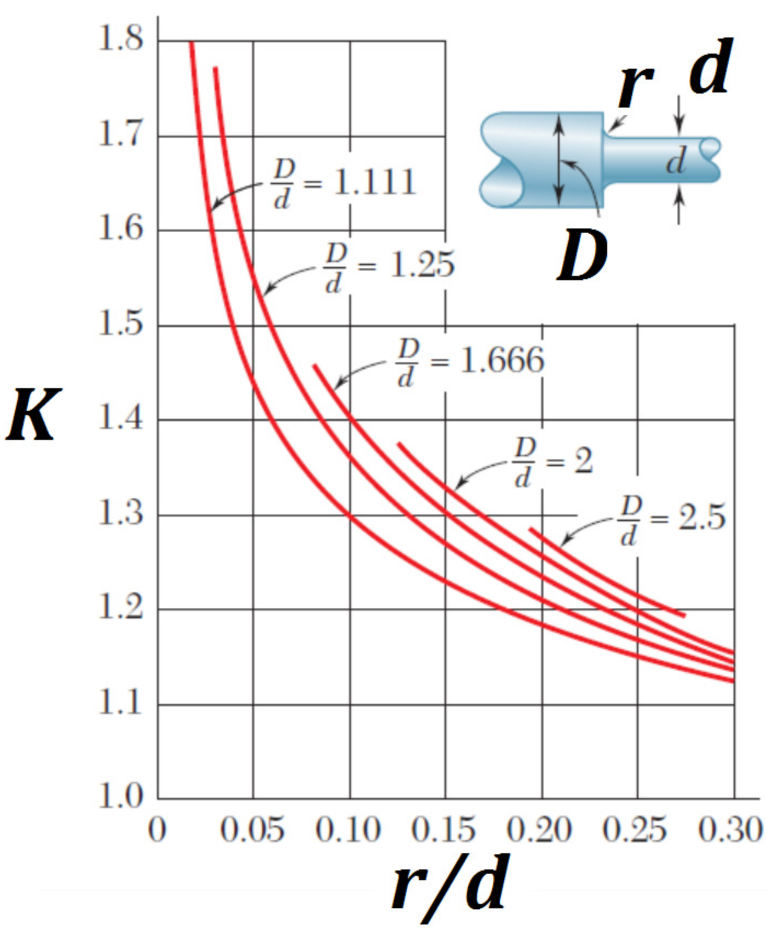
(Watt) = (N.m)(rad/sec)

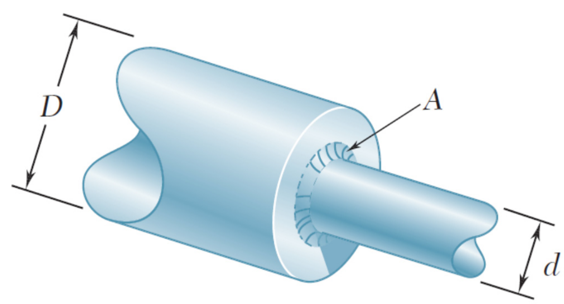
$$\omega = 2\pi f$$

where f is frequency of rotation and its unit is 1/sec or Hz

$$\rightarrow P = 2\pi f T \rightarrow T = \frac{P}{2\pi f} \quad \therefore \tau_{max} = \frac{Tc}{J} \rightarrow T = \frac{J\tau_{max}}{c}$$

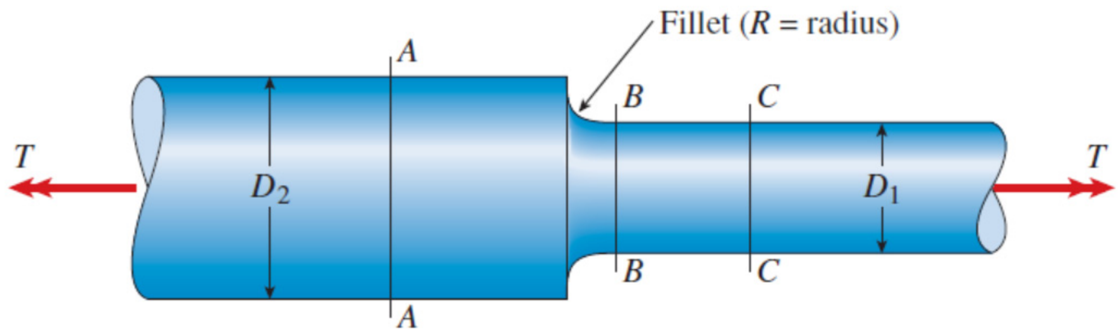
Stress Concentration in Circular Shaft



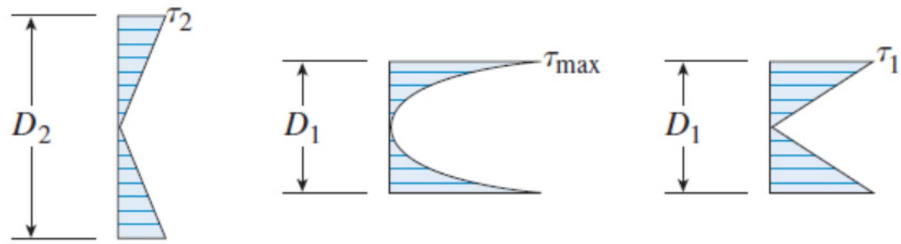


Shaft with change in diameter.

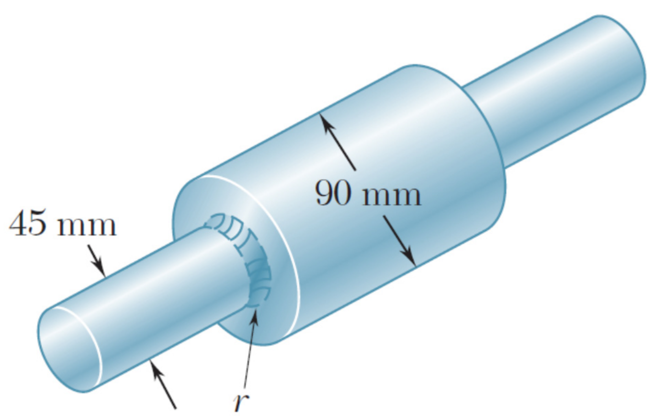
$$K = \frac{\tau_{max}}{\frac{Td/2}{J}}$$



(a)



Example 9: The stepped shaft shown must transmit 40 kW at a speed of 720 rpm. Determine the minimum radius r of the fillet if an allowable stress of 36 MPa is not to be exceeded.



$P = 40\ kW, \omega = 720\ rpm$

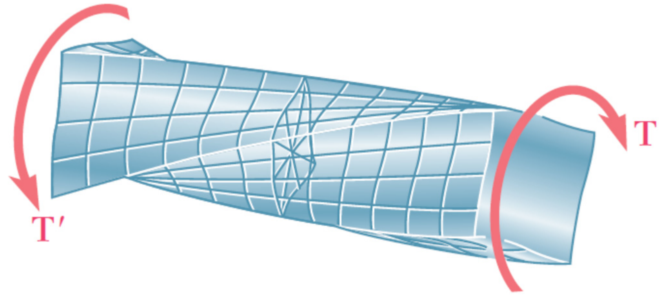
$$\rightarrow T = \frac{P}{\omega} = \frac{40\,000\ Watt}{720 \times \frac{2\pi\ rad}{60\ sec}} = 530.52\ Nm$$

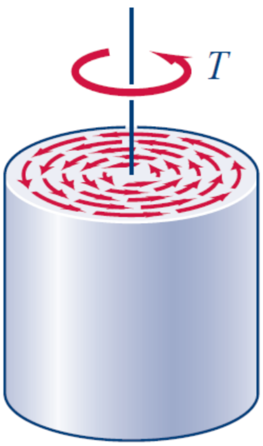
$$K = \frac{\tau_{max}}{\frac{\frac{Td}{2}}{J}} = \frac{36\ MPa}{\frac{(530.52 \times 10^3\ Nmm)\left(\frac{45\ mm}{2}\right)}{\frac{\pi}{2}\left(\frac{45\ mm}{2}\right)^4}} = 1.21 \rightarrow K = 1.21\ \ and\ \ \frac{D}{d} = 2$$

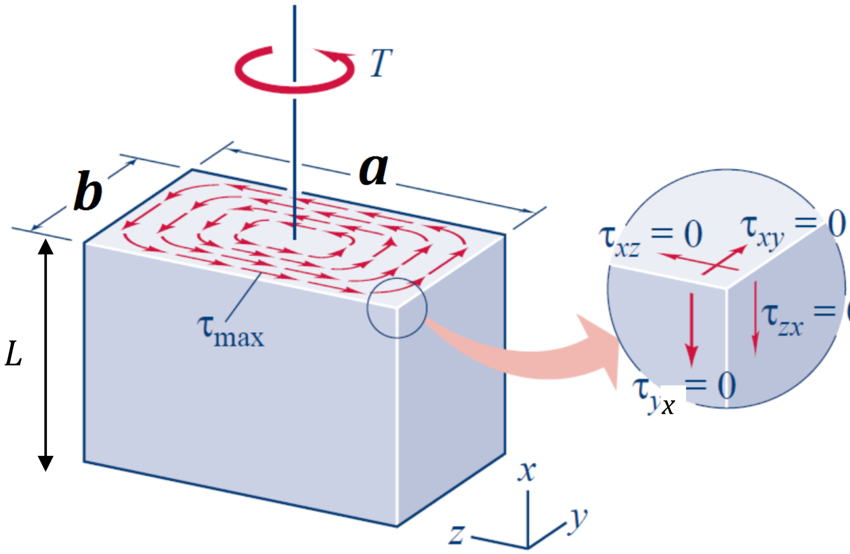
$$\rightarrow from\ the\ graph \rightarrow \frac{r}{d} \cong 0.25 \rightarrow r = 0.25\ d = 0.25\ (45\ mm) = 10.8\ mm$$

Torsion of Noncircular Members

An important feature of the torsional deformation of noncircular prismatic bars is the ***wraping*** of the cross sections. The theory of elasticity may be used to relate the torque applied to such noncircular prismatic members to the resulting stress distribution and angle of twist.

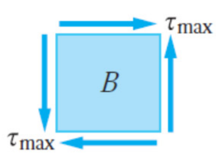


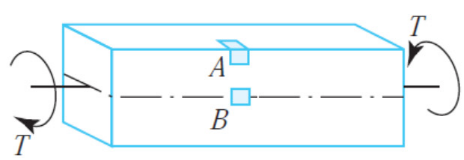


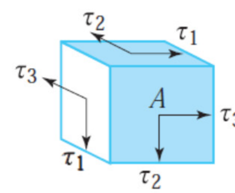


$$\tau_{max} = \frac{T}{c_1 ab^2} \quad and \quad \varphi = \frac{TL}{c_2 ab^3 G}$$

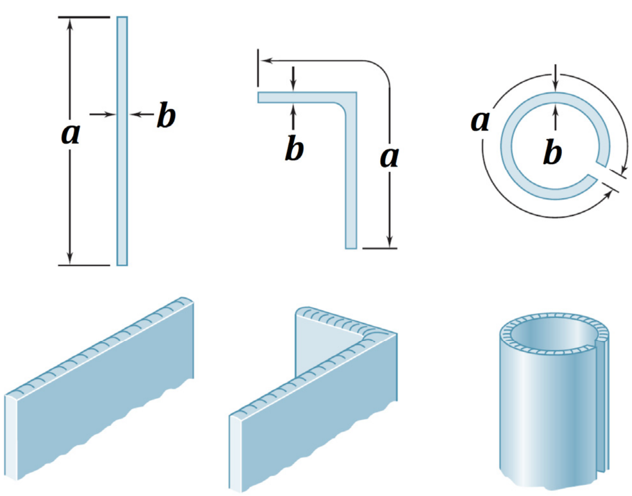
$$c_1 = c_2 = \frac{1}{3}\left(1 - 0.63\frac{b}{a}\right) \quad for \ \frac{a}{b} \geq 5$$







$\tau_1 = \tau_2 = \tau_3 = 0$

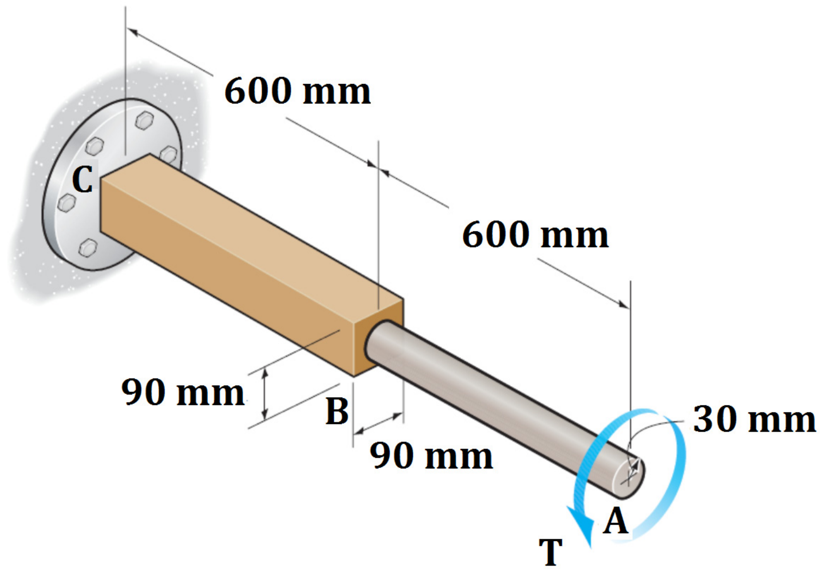


Coefficients for Rectangular Bars in Torsion

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

For ***thin-walled open-section*** members of ***uniform thickness*** (as those shown above) the same formulation can be used to determine maximal stress and angle of twist. Maximal stress is approximately the same over the long side surfaces except in the vicinity of the short sides.

Example 10: Segments AB and BC of the shaft have circular and square cross sections, respectively. The shaft is made from A-36 steel (G = 75 GPa) with an allowable shear stress of $\tau_{all} = 75\ MPa$ and an angle of twist at end A which is not allowed to exceed 0.02 rad. Determine the maximum allowable torque **T** that can be applied at end A. The shaft is fixed at C.



Maximum shear stress in shaft AB:

$$\tau_{max_{AB}} = \frac{T_{AB} r_{AB}}{J_{AB}} \rightarrow 75\ MPa = \frac{T\ (30\ mm)}{\frac{\pi}{2} 30^4\ mm^4} \rightarrow T = 3\ 180\ 860\ Nmm$$

$$\rightarrow T = 3.18\ kNm$$

Maximum shear stress in shaft BC:

$$\tau_{max_{BC}} = \frac{T_{BC}}{c_1 ab^2}, \quad a = b = 90\ mm \rightarrow \frac{a}{b} = 1 \rightarrow c_1 = 0.208\ \ and\ \ c_2 = 0.1406$$

$$\rightarrow \tau_{max_{BC}} = \frac{T_{BC}}{c_1 ab^2} \rightarrow 75\ MPa = \frac{T}{0.208 \times 90\ mm \times 90^2\ mm^2}$$

$$\rightarrow T = 11\ 372\ 400\ Nmm = 11.37\ kNm$$

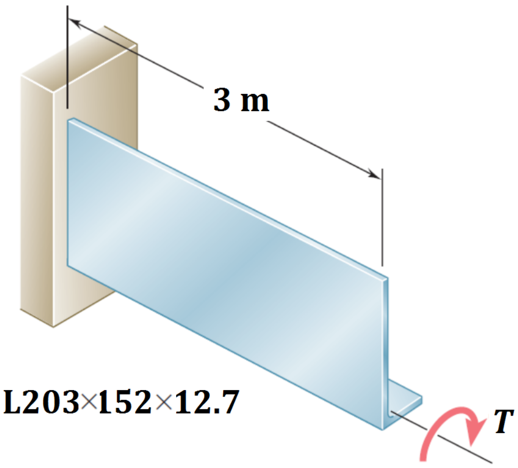
Maximum angle of twist at A:

$$\varphi_A = \varphi_{A/B} + \varphi_{B/c} + \varphi_C \rightarrow 0.02 = \frac{T_{AB} L_{AB}}{J_{AB} G} + \frac{T_{BC} L_{BC}}{c_2 ab^3 G} + 0$$

$$\rightarrow 0.02 = \frac{T\ (600\ mm)}{\frac{\pi}{2} 30^4\ mm^4 \times 75\ 000\ MPa} + \frac{T\ (600\ mm)}{0.1406 \times 90\ mm \times 90^3 mm^3 \times 75\ 000\ MPa}$$

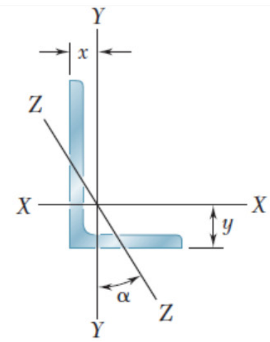
$$\rightarrow T = 2\ 795\ 311\ Nmm \rightarrow T = 2.79\ kNm\ \ (\boldsymbol{controls})$$

Example 11: A 3-m-long steel angle has an L203 × 152 × 12.7 cross section. From Table below we find that the thickness of the section is 12.7 mm and that its area is 4350 mm². Knowing that τ_{all} = 50 MPa and that G = 77.2 GPa, and ignoring the effect of stress concentrations, determine (a) the largest torque T that can be applied, (b) the corresponding angle of twist.



APPENDIX C Properties of Rolled-Steel Shapes
(SI Units)

Angles
Unequal Legs



Size and Thickness, mm	Mass per Meter kg/m	Area mm²	Axis X-X: I_x 10^6 mm⁴	Axis X-X: S_x 10^3 mm³	Axis X-X: r_x mm	Axis X-X: y mm	Axis Y-Y: I_y 10^6 mm⁴	Axis Y-Y: S_y 10^3 mm³	Axis Y-Y: r_y mm	Axis Y-Y: x mm	Axis Z-Z: r_z mm	Axis Z-Z: tan α
L203 × 152 × 25.4	65.5	8390	33.7	247	63.2	67.3	16.1	146	43.7	41.9	32.5	0.542
19	50.1	6410	26.4	192	64.0	64.8	12.8	113	44.5	39.6	32.8	0.550
12.7	34.1	4350	18.5	131	64.8	62.5	9.03	78.5	45.5	37.1	33.0	0.557

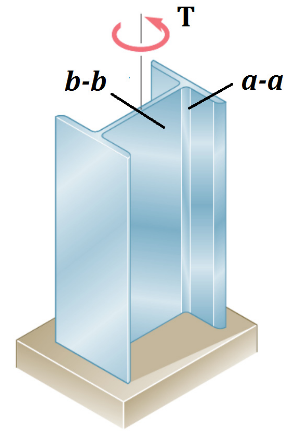
$$\tau_{max} = \frac{T}{c_1 ab^2} \rightarrow 50\ MPa = \frac{T}{c_1 ab^2}$$

$$b = 12.7\ mm \quad and \quad A = ab \rightarrow 4350\ mm^2 = a\ (12.7\ mm) \rightarrow a = 342.52\ mm$$

$$\rightarrow \frac{a}{b} = \frac{342.52}{12.7} = 26.97 > 5 \rightarrow c_1 = c_2 = \frac{1}{3}\left(1 - 0.63\frac{b}{a}\right) = \frac{1}{3}\left(1 - 0.63\frac{12.7\ mm}{342.52\ mm}\right) = 0.325$$

$$\rightarrow 50\ MPa = \frac{T}{c_1 ab^2} \rightarrow 50\ MPa = \frac{T}{0.325 \times 342.52\ mm \times 12.7^2\ mm^2} \rightarrow T = 899\ 242\ Nmm$$

$$\varphi = \frac{TL}{c_2 ab^3 G} = \frac{899242\ Nmm\ \times 3000\ mm}{0.325 \times 342.52\ mm \times 12.7^3\ mm^3 \times 77\ 200\ MPa} = 0.1532^{rad} = 8.78°$$

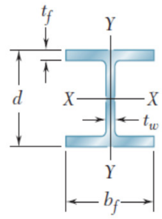


W 200 × 46.1

Example 12: A 2.4-m-long steel member (G = 77 GPa) has a W200×46.1 cross section. Knowing that T = 560 Nm determine (a) maximum shearing stress along lines a-a and b-b as well as the angle of twist.

APPENDIX C Properties of Rolled-Steel Shapes
(SI Units)
Continued from page A18

W Shapes
(Wide-Flange Shapes)



Designation†	Area A, mm^2	Depth d, mm	Flange Width b_f, mm	Flange Thickness t_f, mm	Web Thickness t_w, mm	Axis X-X I_x 10^6 mm^4	Axis X-X S_x 10^3 mm^3	Axis X-X r_x mm	Axis Y-Y I_y 10^6 mm^4	Axis Y-Y S_y 10^3 mm^3	Axis Y-Y r_y mm
W200 × 86	11000	222	209	20.6	13.0	94.9	852	92.7	31.3	300	53.3
71	9100	216	206	17.4	10.2	76.6	708	91.7	25.3	246	52.8
59	7550	210	205	14.2	9.14	60.8	582	89.7	20.4	200	51.8
52	6650	206	204	12.6	7.87	52.9	511	89.2	17.7	174	51.6
46.1	5880	203	203	11.0	7.24	45.8	451	88.1	15.4	152	51.3
41.7	5320	205	166	11.8	7.24	40.8	398	87.6	9.03	109	41.1
35.9	4570	201	165	10.2	6.22	34.4	342	86.9	7.62	92.3	40.9

From Appendix C for the flanges: b = 11 mm and a = 203 mm so we have a/b = 18.45 > 5

$$c_1 = c_2 = \frac{1}{3}\left(1 - 0.63\frac{b}{a}\right) = \frac{1}{3}\left(1 - 0.63\frac{11\ mm}{203\ mm}\right) = 0.322$$

$$T_w + 2T_f = 560\ Nm$$

From Appendix C for the web: b = 7.2 mm and $a = d\text{-}2\,t_f$ =203-2(11) = 181 mm, a/b = 25.14 > 5

$$c_1 = c_2 = \frac{1}{3}\left(1 - 0.63\frac{b}{a}\right) = \frac{1}{3}\left(1 - 0.63\frac{7.2\ mm}{181\ mm}\right) = 0.325$$

The system is statically indeterminate so we need a compatibility equation: $\varphi_f = \varphi_w \rightarrow$

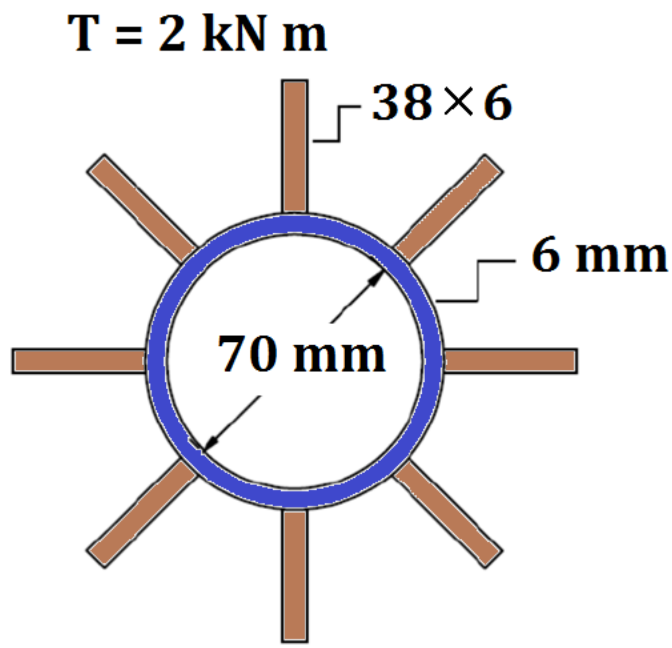
$$\frac{T_f L_f}{(c_2 ab^3 G)_f} = \frac{T_w L_w}{(c_2 ab^3 G)_w} \rightarrow T_f = 3.96T_w \rightarrow T_f = 248.61\ Nm\ and\ T_w = 62.78\ Nm$$

$$\tau_f = \frac{T_f}{(c_1 ab^2)_f} = \frac{248.61 \times 10^3\ Nmm}{0.322 \times 203\ mm \times 11^2\ mm^2} = 31.43\ MPa$$

$$\tau_w = \frac{T_w}{(c_1 ab^2)_w} = \frac{62.78 \times 10^3\ Nmm}{0.325 \times 181\ mm \times 7.2^2\ mm^2} = 20.58\ MPa$$

$$\varphi_f = \varphi_w = 0.089^{rad} = 5.1°$$

Example 13: A hollow tube with radial fins is subjected to a torque T = 2 kNm. Find the torque transmitted to the fins and the maximum shear stress.

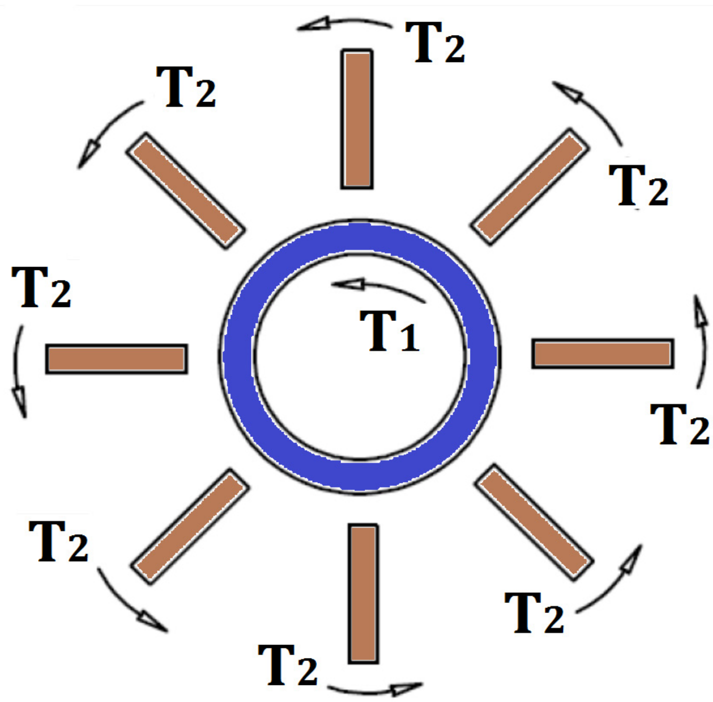


$T = 2\,kNm = T_1 + 8\,T_2 \quad (1)$
$\rightarrow$ *statically indeterminate*

Compatibility equation:

$$\varphi_{tube} = \varphi_{fins}$$
$$\frac{T_1 L}{J_1 G} = \frac{T_2 L}{c_2 a b^3 G} \quad where\ \frac{a}{b} = \frac{38}{6} > 5$$

$$c_1 = c_2 = \frac{1}{3}\left(1 - 0.63\frac{b}{a}\right)$$
$$= \frac{1}{3}\left(1 - 0.63\frac{6\,mm}{38\,mm}\right)$$
$$= 0.300$$
$$\frac{T_1 L}{J_1 G} = \frac{T_2 L}{c_2 a b^3 G} \rightarrow \frac{T_1}{\frac{\pi}{2}(41^4 - 35^4)}$$
$$= \frac{T_2}{0.3 \times 38 \times 6^3} \rightarrow$$
$$T_1 = 845.3\,T_2 \quad (2)$$
$$T_2 = 2.34\,Nm, T_1 = 1981.25\,Nm$$



The fins carry less than 1% of the torque.

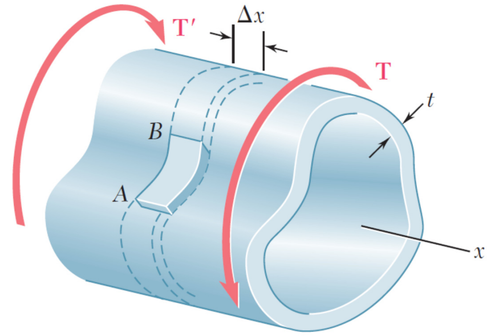
$$\tau_{tube} = \frac{T_1 r}{J}$$
$$= \frac{(1981.25 \times 10^3\,Nmm)(41\,mm)}{\frac{\pi}{2}(41^4 - 35^4)}$$
$$= 39\,MPa$$

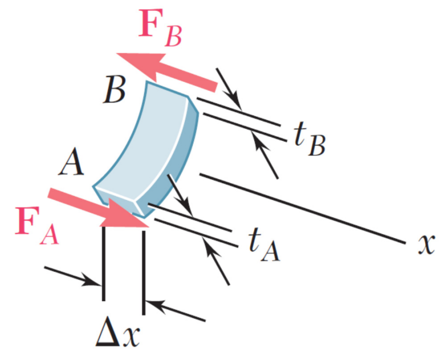
$$\tau_{fin} = \frac{T_2}{c_1 a b^2} = \frac{2.34 \times 10^3 Nmm}{0.3 \times 38 \times 6^2}$$
$$= 5.71\,MPa$$

$$\rightarrow \tau_{max} = 39\,MPa$$

Thin-walled Hollow Shafts (noncircular closed section)

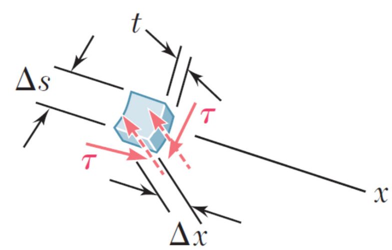
As indicated the determination of stresses in noncircular members generally requires the use of advanced mathematical methods. In the case of thin-walled hollow noncircular shafts (such as light-weighted frameworks in aircrafts and spacecraft), however, a good approximation of the distribution of stresses in the shaft can be obtained by a simple computation. Consider a hollow ***prismatic***[1] cylindrical ***(cross section does not vary along the length of the member)*** member of *noncircular* **closed**[2] section subjected to a torsional loading. While the thickness t of the wall ***may vary within a transverse section***[3], it will be assumed that ***it remains small compared to the other dimensions***[4] of the member. As t is small we can assume that ***shear stress remains constant through wall thickness***[5]. We now detach from the member the portion of wall AB bounded by two transverse planes at a distance Δx from each other, and by two longitudinal planes perpendicular to the wall. Considering equilibrium:





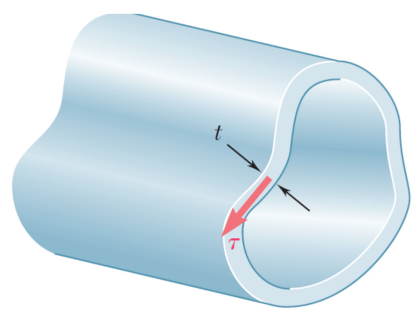
$$F_A = F_B \rightarrow \tau_A(t_A\Delta x) = \tau_B(t_B\Delta x) \rightarrow \tau_A t_A = \tau_B t_B = \tau t = constant = q$$

q is called ***shear flow.***



Small element from segment

We now detach a small element from the wall portion AB. Since the upper and lower faces of this element are part of the free surface of the hollow member, the stresses on these faces are equal to zero. It follows that the stress components indicated on the other faces by dashed arrows are also zero, while those represented by solid arrows are equal. ***Thus, the shearing stress at any point of a transverse section of the hollow member is parallel to the wall surface***[6]***.***



Direction of shearing stress on cross section

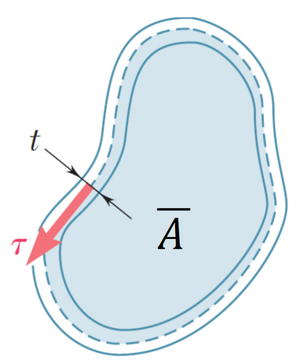
We will now derive a relation between the torque T applied to a hollow member and the shear flow q in its wall:

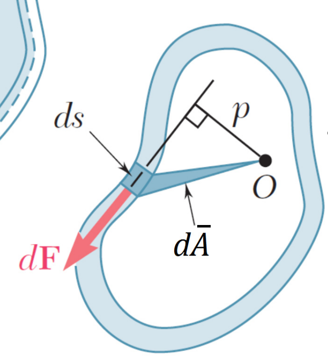
$$dF = \tau\, dA = \tau(t \times ds) = (\tau t)ds = q\, ds$$

$$dT = pdF = p\,(q\, ds) = q(p\, ds) = q(2d\bar{A})$$

$$T = \oint dT = \oint q(2d\bar{A}) = 2q\bar{A} \rightarrow q = \frac{T}{2\bar{A}}$$

$$\tau = \frac{q}{t} \rightarrow \boldsymbol{\tau_{avg}} = \frac{\boldsymbol{T}}{\boldsymbol{2\bar{A}t}}$$





O is an arbitrary point

This is average shear stress as it is based on assumption that shear stress does not vary across wall thickness. ***Maximal shear stress occurs where t is minimal.***

$\boldsymbol{\tau_{avg}}$ = the average shear stress acting over a particular thickness of the tube. We call it average stress as we assumed that shear stress remains constant through wall thickness.
$\boldsymbol{T}$ = the resultant internal torque at the cross section
$\boldsymbol{t}$ = the thickness of the tube where $\boldsymbol{\tau_{avg}}$ is to be determined
$\boldsymbol{\bar{A}}$ = the mean area enclosed within the boundary of the *centerline* of the tube's thickness.

Angle of Twist

The angle of twist of a thin-walled tube of length L can be determined using energy methods (we will see it in Strength of Materials II). If the material behaves in a linear elastic manner and G is the shear modulus, then this angle ($\boldsymbol{\varphi}$), given in radians, can be expressed as:

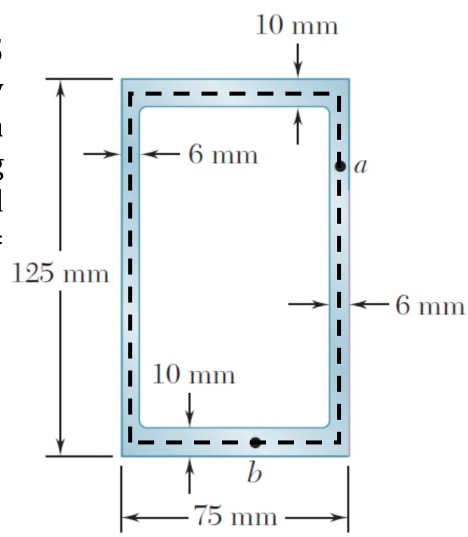
$$\boldsymbol{\varphi = \frac{TL}{4\bar{A}^2 G}\oint \frac{ds}{t}}$$

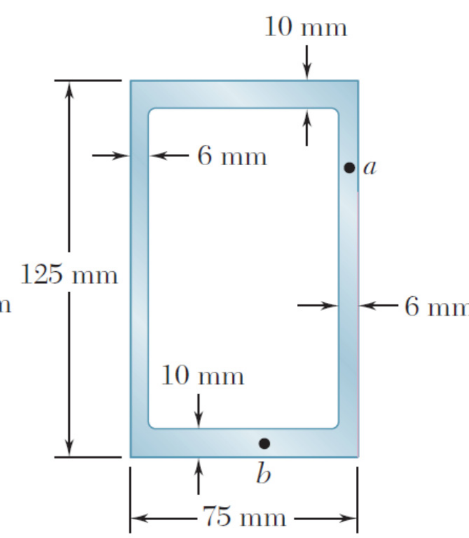
Here the integration must be performed around the entire boundary of the tube's cross-sectional area (see the example below). If $\boldsymbol{t}$ remains constant through the section then we can write:

$$\boldsymbol{\varphi = \frac{TL}{4\bar{A}^2 G}\oint \frac{ds}{t} = \frac{TL}{4\bar{A}^2 Gt}\oint ds = \frac{TLS}{4\bar{A}^2 Gt}}$$

where S is the length of the centerline.

Example 14: A torque T = 5 kNm is applied to a hollow shaft having the cross section shown. Determine the shearing stress at points a and b. Find angle of twist if L = 2 m and G = 77 GPa.





$$\bar{A} = (125 - 2 \times 5) \times (75 - 2 \times 3) = 7935\ mm^2$$

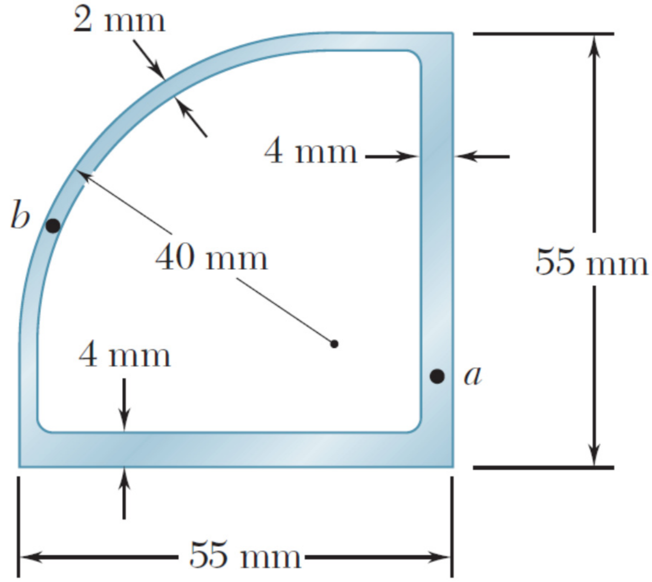
$$\tau_a = \frac{T}{2\bar{A}t_a} = \frac{5\ 000\ 000\ Nmm}{2 \times 7935\ mm^2 \times 6\ mm} = 52.5\ MPa$$

$$\tau_b = \frac{T}{2\bar{A}t_b} = \frac{5\ 000\ 000\ Nmm}{2 \times 7935\ mm^2 \times 10\ mm} = 31.5\ MPa$$

$$\varphi = \frac{TL}{4\bar{A}^2 G}\oint \frac{ds}{t} = \frac{(5 \times 10^6\ Nmm)(2000\ mm)}{4(7935\ mm^2)^2(77000\ MPa)}\left(2 \times \frac{125-10}{6} + 2 \times \frac{75-6}{10}\right) = 0.0269\ rad = 1.5°$$

(71)

Example 15: A 90-Nm torque is applied to a hollow shaft having the cross section shown. Determine the shearing stress at points *a* and *b*.

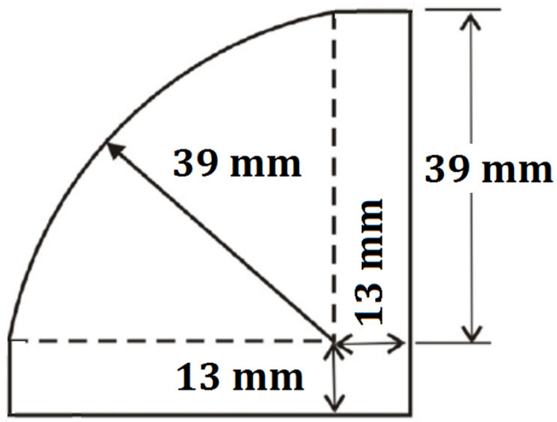


$$\bar{A} = 2 \times 39 \times 13 + 13 \times 13 + \frac{\pi}{4}39^2 = 2377.6\ mm^2$$

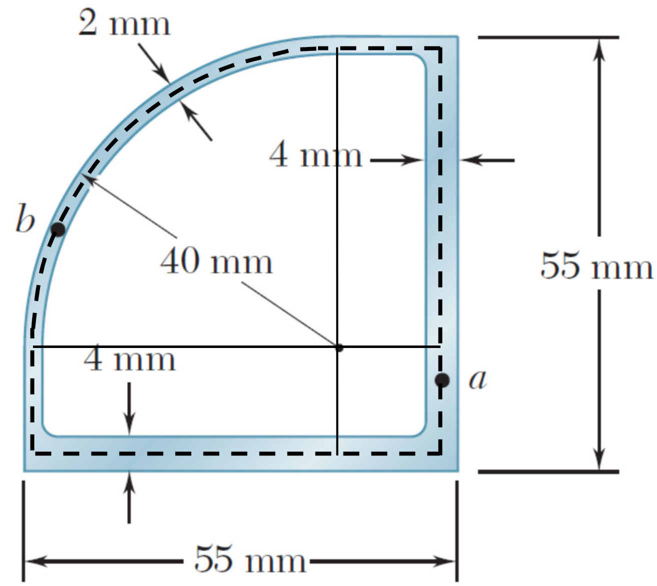
$$\tau_a = \frac{T}{2\bar{A}t_a} = \frac{90\ 000\ Nmm}{2 \times 2377.6\ mm^2 \times 4\ mm} = 4.73\ MPa$$

$$\tau_b = \frac{T}{2\bar{A}t_b} = \frac{90\ 000\ Nmm}{2 \times 2377.6\ mm^2 \times 2\ mm} = 9.46\ MPa$$

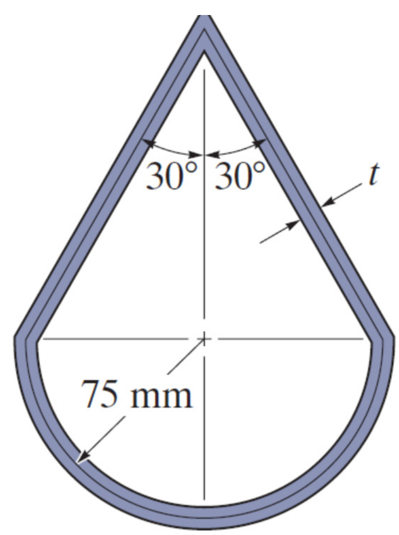
$$\varphi = \frac{TL}{4\bar{A}^2G}\oint\frac{ds}{t} = \frac{TL}{4\bar{A}^2G}[2\left(\frac{52}{4}\right) + 2\left(\frac{13}{2}\right) + \frac{2\pi(39)/4}{2}]$$



Center line



TBR 5 (1390): A shaft has the cross section shown and is made of 2014-T6 aluminium alloy (G = 27 GPa) having an allowable shear stress of τ_{all}= 125 MPa. If the angle of twist per meter length is not allowed to exceed 0.03 rad, determine the required minimum wall thickness *t* when the shaft is subjected to a torque of T = 15 kNm.



$$\bar{A} = \frac{75}{\tan 30°} \times \frac{150}{2} + \frac{\pi}{2}75^2 = 18578.51\ mm^2$$

$$\tau = \frac{T}{2\bar{A}t_a} \rightarrow 125\ MPa = \frac{15\ 000\ 000\ Nmm}{2 \times 18578.51\ mm^2 \times t}$$

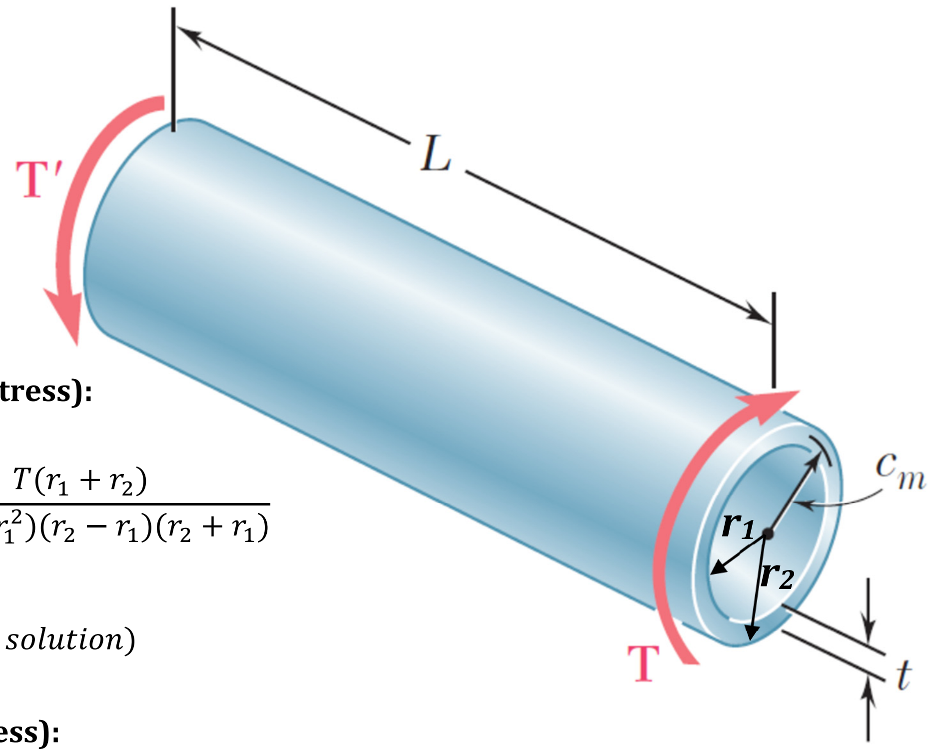
$$\rightarrow t = 3.22\ mm$$

$$\varphi = \frac{TL}{4\bar{A}^2G}\oint\frac{ds}{t}$$

$$\rightarrow \frac{\varphi}{L} = \frac{T}{4\bar{A}^2G}\oint\frac{ds}{t} \rightarrow \frac{0.03}{1000\ mm} = \frac{15\ 000\ 000\ Nmm}{4(18578.51\ mm^2)^2(27\ 000\ MPa)}\left(\frac{\frac{2 \times 75\ mm}{\sin 30°} + \pi \times 75\ mm}{t}\right)$$

$$\rightarrow t = 7.18\ mm\ (controls)$$

Example 16: Calculate shear stress at mid-thickness and the angle of twist from elastic torsion formula and once from thin-walled theory.



Elastic torsion formula (stress):

$$\tau_1 = \frac{T\left(\frac{r_1+r_2}{2}\right)}{\frac{\pi}{2}(r_2^4 - r_1^4)} = \frac{T(r_1+r_2)}{\pi(r_2^2+r_1^2)(r_2-r_1)(r_2+r_1)}$$

$$\rightarrow \tau_1 = \frac{T}{\pi(r_2^2+r_1^2)t} \ (exact\ solution)$$

Thin-walled formula (stress):

$$\tau_2 = \frac{T}{2\bar{A}t} = \frac{T}{2\pi\left(\frac{r_1+r_2}{2}\right)^2 t} = \frac{2T}{\pi(r_1+r_2)^2 t} \ (not\ the\ exact\ solution)$$

If the shaft is thin-walled we have: $r_1 = r_2$:

$$\tau_1 = \frac{T}{\pi(r_2^2+r_1^2)t} = \frac{T}{2\pi r_1^2 t} \ and\ \tau_2 = \frac{2T}{\pi(r_1+r_2)^2 t} = \frac{2T}{\pi(4r_1^2)t} = \frac{T}{2\pi r_1^2 t} \rightarrow \tau_1 = \tau_2$$

For very thin-walled shafts the thin-walled formula gives the exact solution and becomes equal to the elastic formula

Elastic torsion formula (angle of twist):

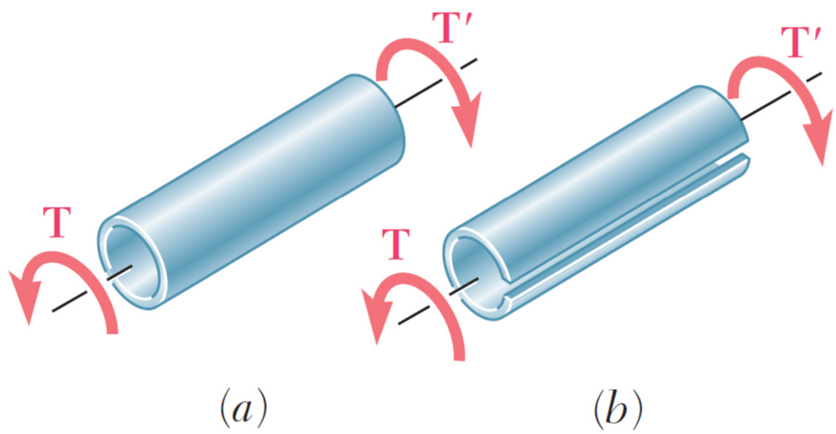
$$\varphi_1 = \frac{TL}{JG} = \frac{TL}{\frac{\pi}{2}(r_2^4 - r_1^4)G} = \frac{2TL}{\pi(r_2^2+r_1^2)(r_2-r_1)(r_2+r_1)G} = \frac{2TL}{\pi(r_2^2+r_1^2)(r_2+r_1)tG}$$

$$\varphi_2 = \frac{TL}{4\bar{A}^2 G}\oint\frac{ds}{t} = \frac{TL}{4\pi^2\left(\frac{r_1+r_2}{2}\right)^4 G}\left(\frac{2\pi\left(\frac{r_1+r_2}{2}\right)}{t}\right) = \frac{4TL}{\pi(r_1+r_2)^3\, tG}$$

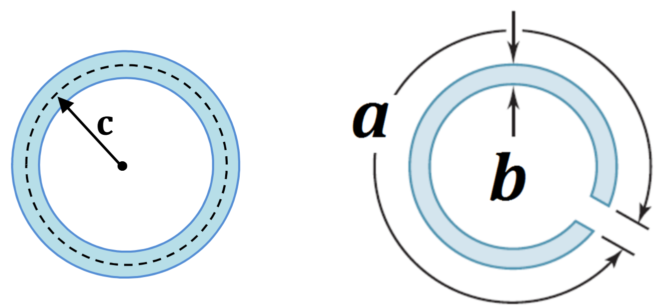
If the shaft is thin-walled we have: $r_1 = r_2$:

$$\varphi_1 = \frac{2TL}{\pi(r_2^2+r_1^2)(r_2+r_1)tG} = \frac{TL}{2\pi r_1^3 tG} \ and\ \varphi_2 = \frac{4TL}{\pi(r_1+r_2)^3\, tG} = \frac{TL}{2\pi r_1^3 tG} \rightarrow \varphi_1 = \varphi_2$$

Example 17: Equal torques are applied to thin-walled tubes of the same length L, same thickness t, and same radius c. One of the tubes has been slit lengthwise as shown. Determine (*a*) the ratio τ_b/τ_a of the maximum shearing stresses in the tubes, (*b*) the ratio φ_b/φ_a of the angles of twist of the tubes.



$$\tau_a = \frac{T}{2\bar{A}t} = \frac{T}{2(\pi c^2)t}$$
$$\tau_b = \frac{T}{c_1 ab^2} = \frac{T}{0.333(2\pi c)t^2}$$
$$\frac{\tau_b}{\tau_a} = \frac{T}{0.333(2c)t^2} \times \frac{2(\pi c^2)t}{T}$$

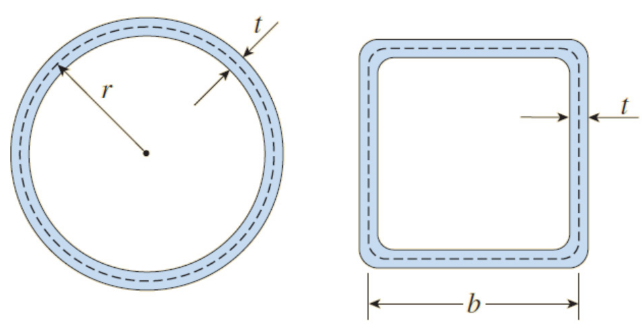


$$\rightarrow \frac{\tau_b}{\tau_a} = \frac{c}{0.333t} = \frac{3c}{t}$$

$$\varphi_A = \frac{TL}{4\bar{A}^2 G}\oint \frac{ds}{t} = \frac{TL}{4(\pi c^2)^2 G}\left(\frac{2\pi c}{t}\right) = \frac{TL}{2\pi c^3 tG}$$

$$\varphi_B = \frac{TL}{c_2 ab^3 G} = \frac{TL}{0.333(2\pi c)t^3 G} = \frac{3TL}{2\pi c t^3 G} \rightarrow \frac{\varphi_B}{\varphi_A} = \frac{3TL}{2\pi c t^3 G} \times \frac{2\pi c^3 tG}{TL} = \frac{3c^2}{t^2}$$

Example 18: A circular tube (1) and a square tube (2) are constructed of the same material and subjected to the same torque. Both tubes have the same length, same wall thickness, and same cross-sectional area. What are the ratios of their shear stresses and angles of twist?



$$2\pi rt = 4bt \rightarrow \pi r = 2b$$
$$\tau_1 = \frac{T}{2\bar{A}t} = \frac{T}{2(\pi r^2)t}$$
$$\tau_2 = \frac{T}{2\bar{A}t} = \frac{T}{2(b^2)t} = \frac{T}{2(\pi^2 r^2/4)t} = \frac{2T}{\pi^2 r^2 t}$$
$$\frac{\tau_1}{\tau_2} = \frac{T}{2\pi r^2 t} \times \frac{\pi^2 r^2 t}{2T} = \frac{\pi}{4} = 0.79$$

$$\varphi_1 = \frac{TL}{4\bar{A}^2 G}\oint \frac{ds}{t} = \frac{TL}{4(\pi r^2)^2 G}\left(\frac{2\pi r}{t}\right) = \frac{TL}{2\pi r^3 tG},\ \varphi_2 = \frac{TL}{4\bar{A}^2 G}\oint \frac{ds}{t} = \frac{TL}{4(b^2)^2 G}\left(\frac{4b}{t}\right) = \frac{TL}{b^3 tG} = \frac{8TL}{\pi^3 r^3 tG}$$

$$\frac{\varphi_1}{\varphi_2} = \frac{TL}{2\pi r^3 tG} \times \frac{\pi^3 r^3 tG}{8TL} = \frac{\pi^2}{16} = 0.62$$

These results show that the circular tube not only has a 21% lower shear stress than does the square tube but also a greater stiffness against rotation.