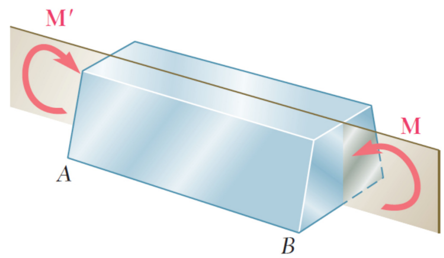
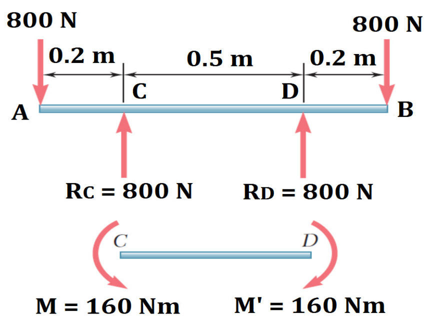
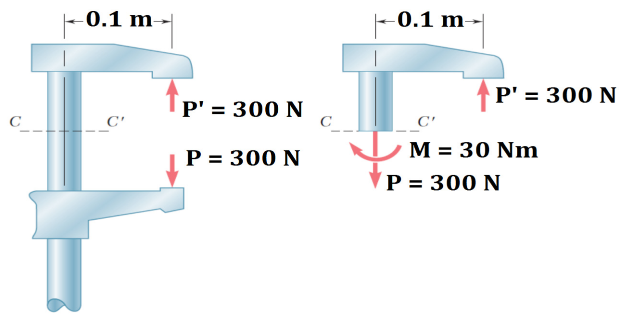
CHAPTER 4: BENDING OF BEAMS



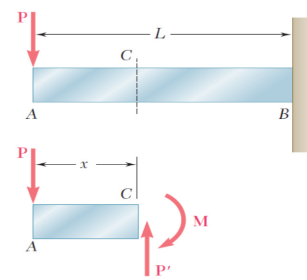
This chapter will be devoted to the analysis of prismatic members subjected to equal and opposite couples **M** and **M'** acting in the same longitudinal plane. Such members are said to be in ***pure bending***. An example of pure bending is provided by the bar of a typical barbell as it is held overhead by a weight lifter as shown. The results obtained for pure bending will be used in the analysis of other types of loadings as well, such as ***eccentric axial loadings*** and ***transverse loadings*** (see examples below).



Pure bending



Bending combined with axial loadings



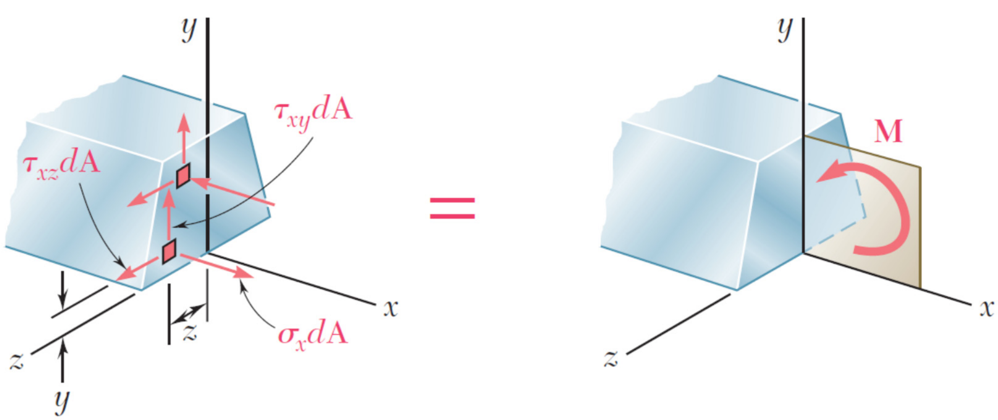
Bending combined with transverse shear

Symmetric Prismatic Member in Pure Bending (Equilibrium)

Assumptions: **(1)** Section has at least one plane of symmetry, **(2)** The bending moment is applied in plane of symmetry and **(3)** the beam is prismatic

Conclusions: **(1)** Only normal stress (uniaxial stress) exists in bending (from theory and experiment), **(2)** There exists a neutral axis, and **(3)** The deflection curve of the bent beam forms a circular arc (from theory and experiment), see below for details.

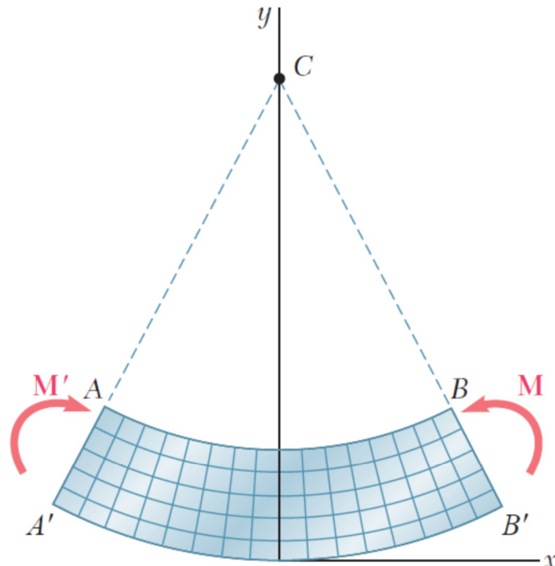
Considering equilibrium: We have total of 3 equilibrium equations as follows:



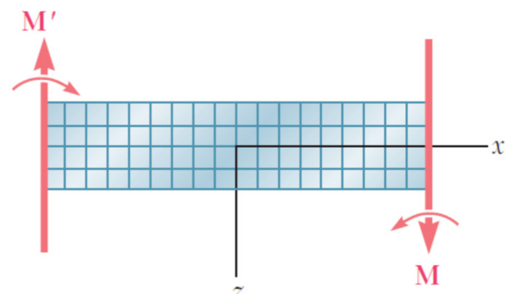
$$\sum F_x = 0 \rightarrow \int \sigma_x \, dA = 0 \;\; (1), \;\; \sum M_y = 0 \rightarrow \int z\sigma_x \, dA = 0 \;\; (2), \sum M_z = 0 \rightarrow \int (y\sigma_x \, dA) + M = 0 \;\; (3)$$

As the variation of σ_x on the section (A) is unknown these equilibrium equations cannot be resolved and therefore the system is statistically indeterminate and we need compatibility.

Deformation of Symmetric Member in Pure Bending



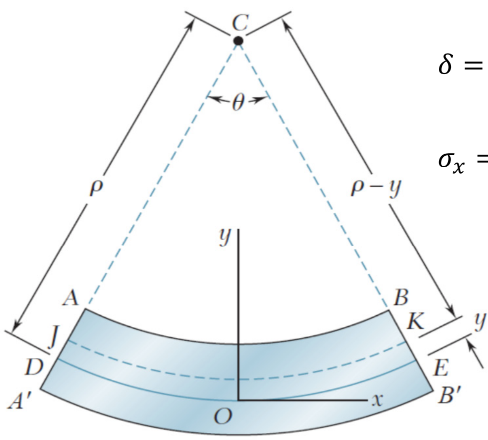
Since all the faces represented in the two projections are at 90° to each other, we conclude that $\gamma_{xy} = \gamma_{zx} = 0$ and, thus, that $\tau_{xy} = \tau_{xz} = 0$. Also, σ_y, σ_z, and τ_{yz}, we note that they must be zero on the surface of the member. Thus, at any point of a slender member in pure bending, we have a state of *uniaxial stress*. Recalling that, for $M > 0$, lines AB and $A'B'$ are observed, respectively, to decrease and increase in length, we note that the strain ε_x and the stress σ_x are negative in the upper portion of the member (*compression*) and positive in the lower portion (*tension*). Therefore there must exist a surface parallel to the upper and lower faces of the member, where ε_x and σ_x are zero. This surface is called the *neutral surface*. **For two reasons it is important to determine position of the neutral axis: 1) to compute maximal stress and 2) to make holds (if needed during design) along the neutral axis (to avoid stress concentration).**



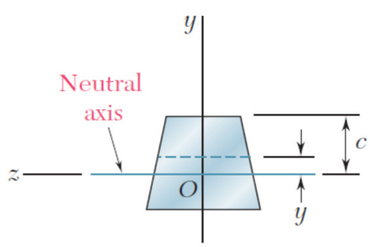
$$L_{DE} = L = \rho\theta,\ \ L_{JK} = L' = (\rho - y)\theta$$

$$\delta = L' - L = -y\theta \rightarrow \varepsilon_x = \frac{\delta}{L} = \frac{-y\theta}{\rho\theta} = -\frac{y}{\rho} \rightarrow |\varepsilon_{max}| = \frac{c}{\rho} \rightarrow \varepsilon_x = -\frac{y}{c}\varepsilon_{max}$$

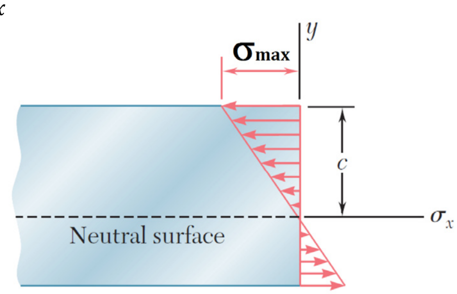
$$\sigma_x = E\varepsilon_x = -\frac{y}{c}(E\varepsilon_{max}) \rightarrow \sigma_x = -\frac{y}{c}\sigma_{max}$$



Longitudinal vertical section



Transverse section

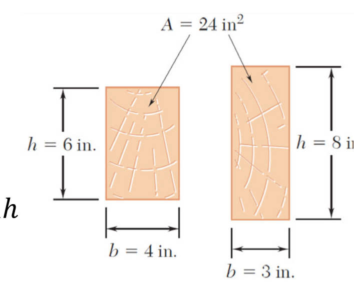


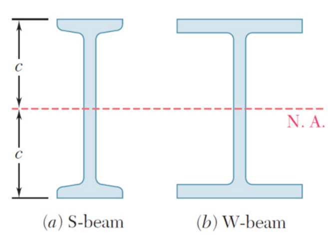
Bending stresses

$\int \sigma_x \, dA = 0 \rightarrow \int -\frac{y}{c}\sigma_{max}\, dA = 0 \rightarrow \int y \, dA = 0 \rightarrow$ Neutral axis passes through the centroid of the section

$$\int (-y\sigma_x \, dA) = M \rightarrow \int \left(\frac{y^2}{c}\sigma_{max}\ dA\right) = M \rightarrow \frac{\sigma_{max}}{c}\int y^2 \, dA = M \rightarrow \frac{\sigma_{max}}{c} I = M \rightarrow \sigma_{max} = \frac{Mc}{I} \rightarrow \boldsymbol{\sigma_x = -\frac{My}{I}}$$

$$\sigma_{max} = \frac{M}{\frac{I}{c}} = \frac{M}{S} \ (S\text{: the elastic section modulus})$$





Optimal design

$$\rightarrow \text{for rectangular section } S = \frac{\frac{1}{12}bh^3}{h/2} = \frac{1}{6}bh^2 = \frac{1}{6}Ah$$

$$\varepsilon_x = -\frac{y}{\rho} \rightarrow \frac{\sigma_x}{E} = -\frac{y}{\rho} \rightarrow \frac{-My}{EI} = -\frac{y}{\rho} \rightarrow \boldsymbol{\frac{1}{\rho} = \frac{M}{EI}} \ (\text{curvature of the neutral axis})$$

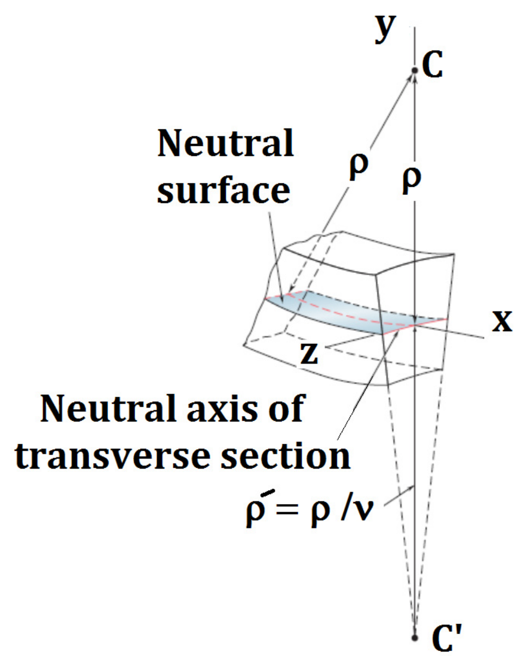
POINT 1: Since in ***pure-bending*** M is constant along the entire length of the beam, deformation of a beam is uniform along the length of the segment undergoing pure bending; so whatever happens at a typical cross-section also happens at any other section. For example, the curvature of the deflection curve at any section is the same as the curvature at any other section. Therefore, **the deflection curve forms a circular arc, with center of curvature at *C*.**

POINT 2: $\int z\sigma_x \, dA = 0 \rightarrow \int -z\frac{y}{c}\sigma_{max} \, dA = 0 \rightarrow \int zy \, dA = 0 \rightarrow I_{yz} = 0 \rightarrow$
$y \text{ and } z \text{ are proncipal axes of the cross section } \rightarrow \text{we already know it from symmetry}$

Deformations in a Transverse Cross Section

As mentioned the transverse cross section of a member in pure bending remains plane but we will have some deformations within the plane of the section.

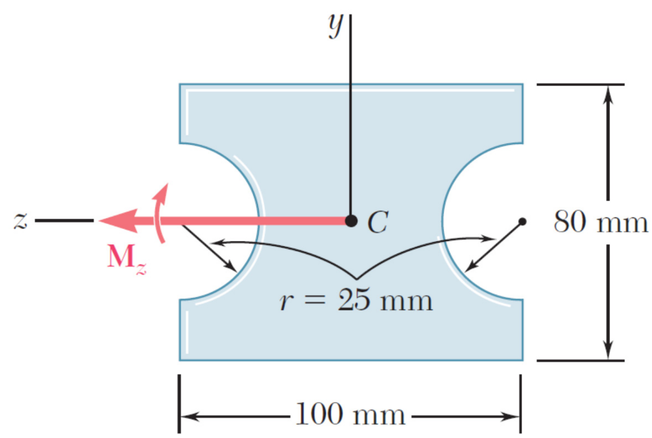
$$\varepsilon_y = -\nu\varepsilon_x, \quad \varepsilon_z = -\nu\varepsilon_x \rightarrow \varepsilon_y = \varepsilon_z = \frac{\nu y}{\rho}$$



The relations we have obtained show that the elements located above the neutral surface (y>0) will expand in both the y and z directions, while the elements located below the neutral surface (y<0) will contract. In the case of a member of rectangular cross section, the expansion and contraction of the various elements in the vertical direction will compensate, and no change in the vertical dimension of the cross section will be observed. As far as the deformations in the horizontal transverse z direction are concerned, however, the expansion of the elements located above the neutral surface and the corresponding contraction of the elements located below that surface will result in the various horizontal lines in the section being bent into arcs of circle.

$$\rho' = \frac{\rho}{\nu} \rightarrow \frac{1}{\rho'} = \frac{\nu}{\rho} \; (Anticlastic \; curvature)$$

Example 1: A nylon spacing bar has the cross section shown. Knowing that the allowable stress for the grade of nylon used is 24 MPa, determine the largest couple M_z that can be applied to the bar.



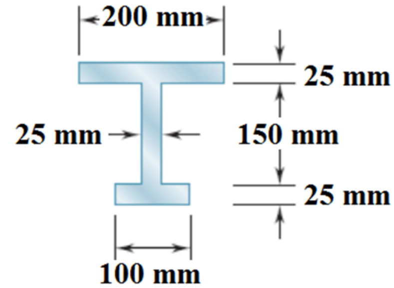
$$\sigma_{max} = \frac{M_z c}{I_z} \rightarrow 24 \, MPa = \frac{M_z \, (40 \, mm)}{I_z}$$

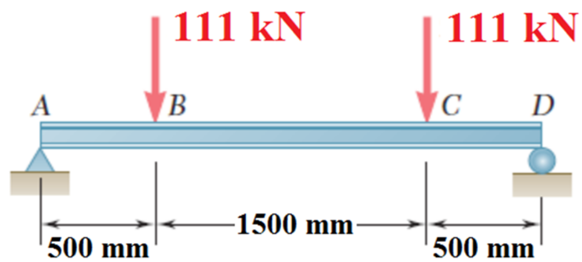
$$I_z = \frac{1}{12}(100 \, mm)(80 \, mm)^3 - \frac{\pi}{4}(25 \, mm)^4 = 3\,959\,871 \, mm^4$$

$$\rightarrow 24 \, MPa = \frac{M_z \, (40 \, mm)}{3959871 \, mm^4} \rightarrow M_z = 2\,375\,922 \, Nmm$$

$$\rightarrow M_z = 2.38 \, kNm$$

Example 2: Two vertical forces are applied to a beam of the cross section shown. Determine the maximum tensile and compressive stresses in portion *BC*.



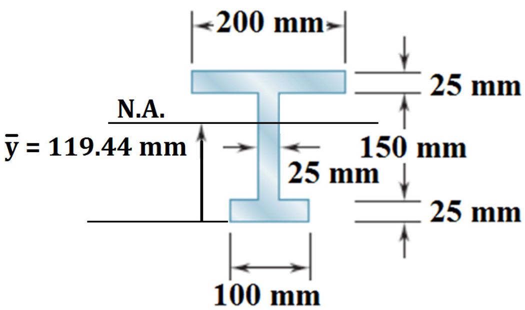


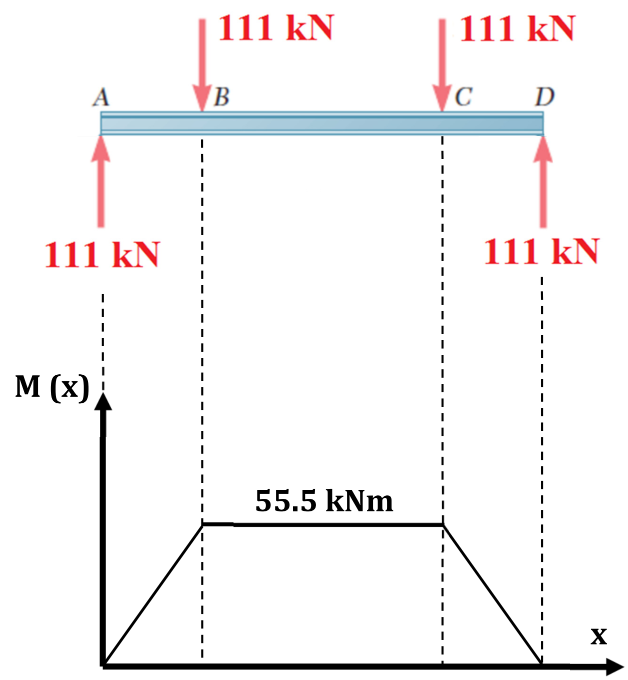
Position of neutral axis (considering origin of the coordinate system at the base):

$$\bar{y} = \frac{\sum \bar{y}A}{\sum A}$$

$$= \frac{12.5 \times (100 \times 25) + 100 \times (150 \times 25) + 187.5 \times (200 \times 25)}{100 \times 25 + 150 \times 25 + 200 \times 25}$$

$$\rightarrow \bar{y} = 119.44\ mm$$





Moment of inertia with respect to the neutral axis:

$$I_{N.A.} = \sum \bar{I} + Ad^2$$

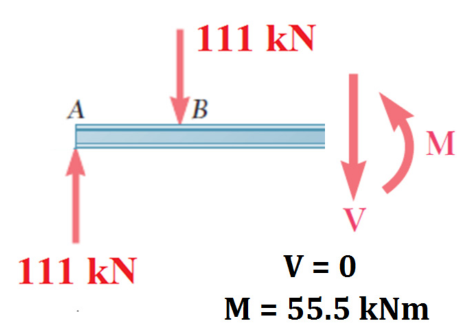
$$= \frac{1}{12}(100)(25^3)$$
$$+ (100 \times 25)(119.44 - 12.5)^2$$
$$+ \frac{1}{12}(25)(150^3)$$
$$+ (25 \times 150)(119.44 - 100)^2$$
$$+ \frac{1}{12}(200)(25^3)$$
$$+ (200 \times 25)(200 - 119.44 - 12.5)^2$$
$$= 60.6 \times 10^6\ mm^4$$

Maximum tensile stress at the lowermost corner:

$$\sigma_{max} = \frac{Mc}{I} = \frac{55.5 \times 10^6 Nmm \times 119.44\ mm}{60.6 \times 10^6\ mm^4} = 109.4\ MPa$$

Maximum compressive stress at the uppermost corner:

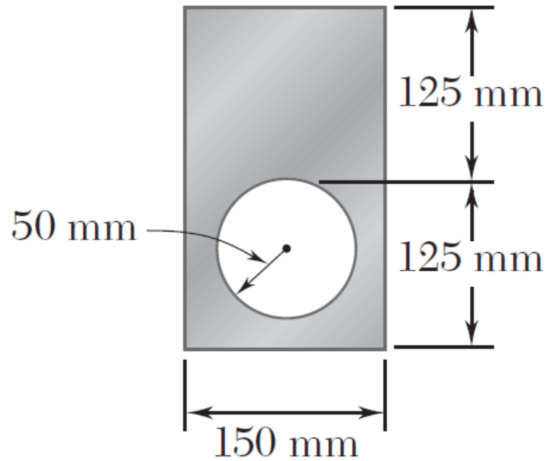
$$\sigma_{max} = \frac{Mc}{I} = -\frac{55.5 \times 10^6 Nmm \times (200 - 119.44)\ mm}{60.6 \times 10^6\ mm^4}$$
$$= -73.8\ MPa$$

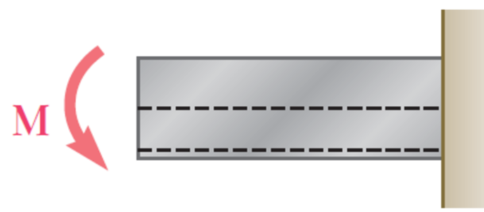


$$\frac{1}{\rho} = \frac{M}{EI} \rightarrow \frac{1}{\rho} = \frac{55\,500\,000\ Nmm}{(200\,000\ MPa)\left(60.6 \times 10^6\ mm^4\right)}$$

$$\rightarrow \rho = 218378.4\ mm = 218.4\ m$$

Example 3: Knowing that for the extruded beam shown the allowable stress is 120 MPa in tension and 150 MPa in compression, determine the largest couple M that can be applied.





Position of neutral axis (considering origin of the coordinate system at the base):

$$\bar{y} = \frac{\sum \bar{y}A}{\sum A}$$

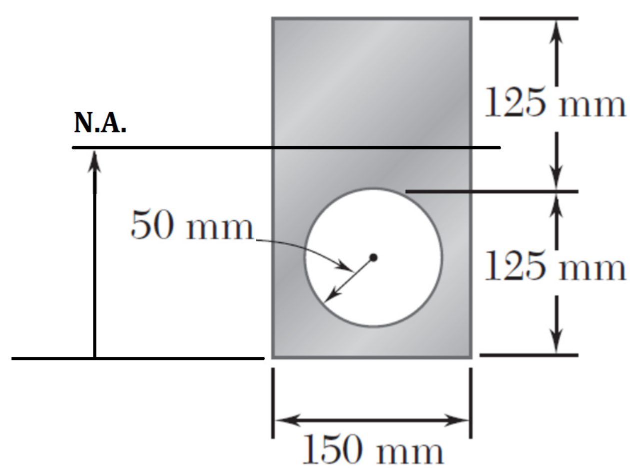
$$= \frac{125 \times (150 \times 250) - (125 - 50) \times (\pi \times 50^2)}{150 \times 250 - \pi \times 50^2}$$

$$\rightarrow \bar{y} = 138.24\ mm$$

Moment of inertia with respect to the neutral axis:

$$I_{N.A.} = \sum \bar{I} + Ad^2 \rightarrow$$

$$I_{N.A.} = \frac{1}{12}(150)(250^3) + (150 \times 250)(125 - 138.24)^2 - \{\frac{\pi}{4} \times 50^4 + \pi \times 50^2 \times (138.24 - 75)^2\} = 165567042\ mm^4$$



Maximum tensile stress at the uppermost corner:

$$\sigma_{max} = \frac{Mc}{I} \rightarrow$$

$$120MPa = \frac{M_{max} \times (250 - 138.24)\ mm}{165567042\ mm^4}$$

$$\rightarrow M_{max} = 177774204\ Nmm = 177.78\ kNm$$ **(Controls)**

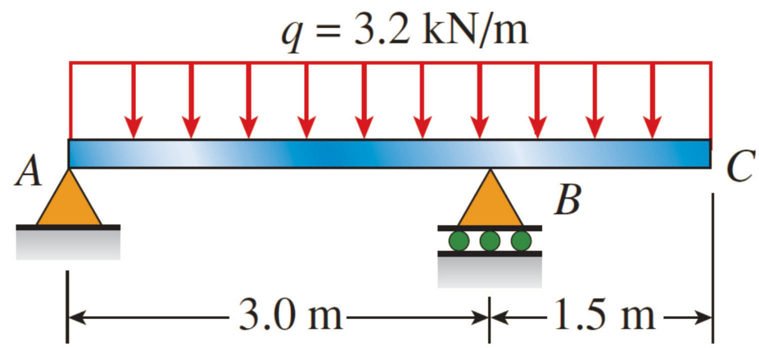
Maximum compressive stress at the uppermost corner:

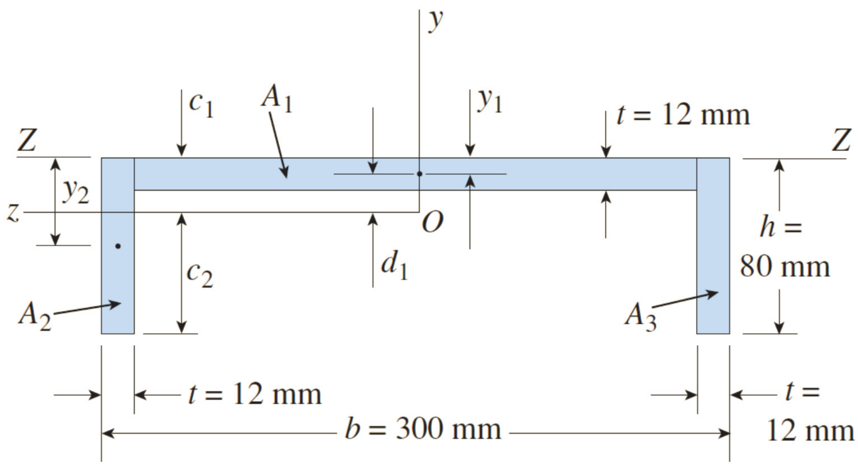
$$\sigma_{max} = \frac{Mc}{I} \rightarrow$$

$$150MPa = \frac{M_{max} \times (138.24)\ mm}{165567042\ mm^4}$$

$$\rightarrow M_{max} = 179651739\ Nmm = 179.65\ kNm$$

Example 4: Determine the maximum tensile and compressive stresses in the beam due to the uniform load (cross section of the beam is shown).





$$c_2 = \frac{2 \times 40 \times (80 \times 12) + 74 \times (12 \times 276)}{2 \times 12 \times 80 + 12 \times 276}$$

$$c_2 = 61.52\ mm$$

$$I_{N.A.} = 2 \times \left(\frac{1}{12} \times 12 \times 80^3 + 12 \times 80 \times 21.52^2\right) + \frac{1}{12} \times 276 \times 12^3 + 276 \times 12 \times 12.48^2 = 2.469 \times 10^6\ mm^4$$

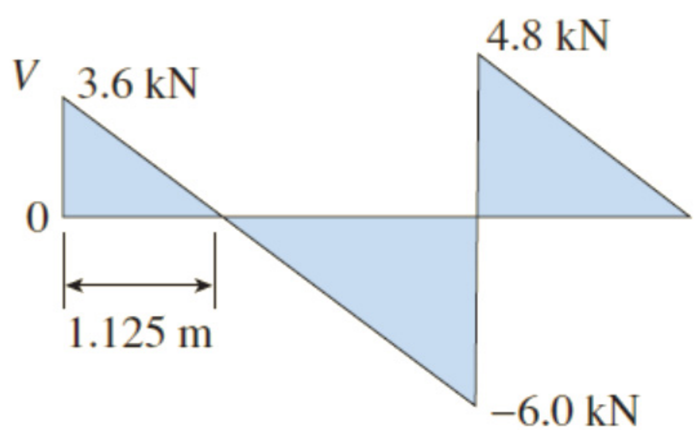
From Statics:

$$M_{max1} = 2.025\ kNm, \quad M_{max2} = 3.6\ kNm$$

Bending stresses due to M_{max1}:

$$\sigma_{\max(tensile)} = \frac{2.025 \times 10^6\ Nmm \times 61.52\ mm}{2.469 \times 10^6\ mm^4} = 50.5\ MPa$$

$$\sigma_{\max(comp)} = -\frac{2.025 \times 10^6\ Nmm \times 18.48\ mm}{2.469 \times 10^6\ mm^4} = -15.2\ MPa$$

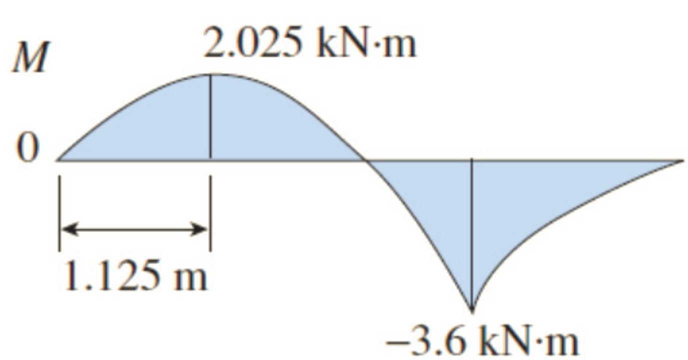


(b)

Bending stresses due to M_{max2}:

$$\sigma_{\max(tensile)} = \frac{3.6 \times 10^6\ Nmm \times 18.48\ mm}{2.469 \times 10^6\ mm^4} = 26.9\ MPa$$

$$\sigma_{\max(comp)} = -\frac{3.6 \times 10^6\ Nmm \times 61.52\ mm}{2.469 \times 10^6\ mm^4} = -89.7\ MPa$$

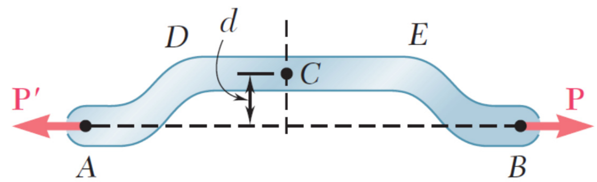


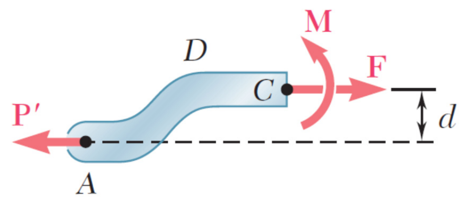
$$\rightarrow \sigma_{\max(tensile)} = 50.5\ MPa$$

$$\rightarrow \sigma_{\max(comp)} = -89.7\ MPa$$

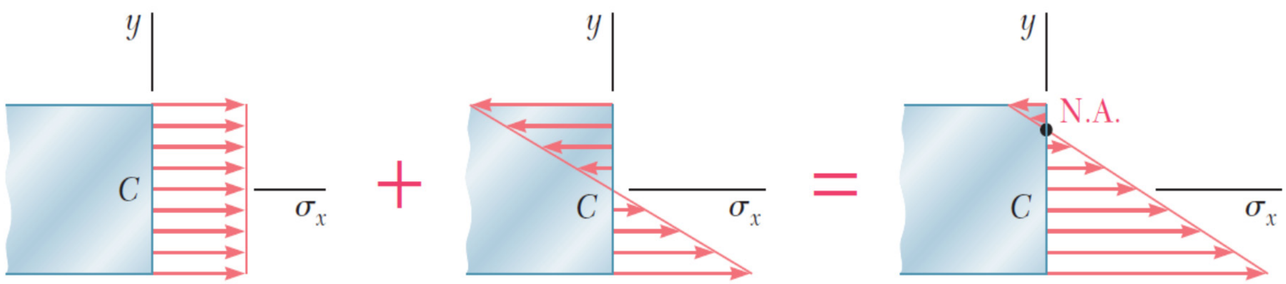
Eccentric Axial Loading in a Plane of Symmetry

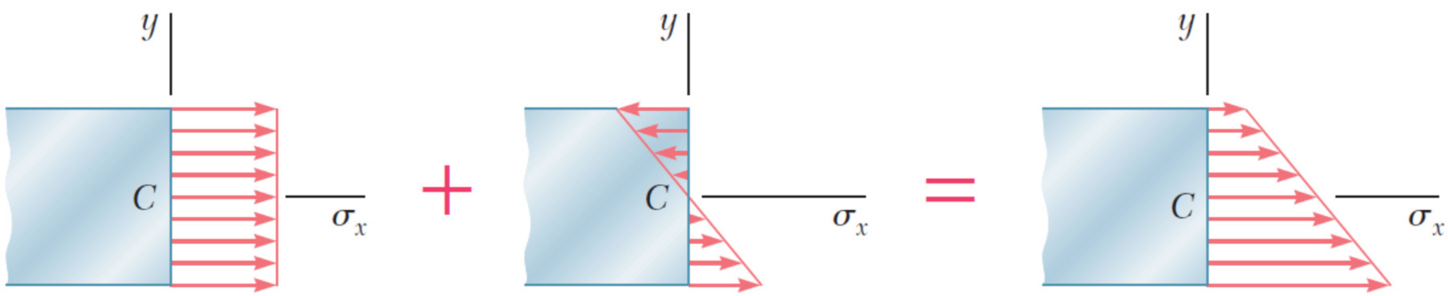
We now analyze the distribution of stresses when the line of action of the loads does *not* pass through the centroid of the cross section, i.e., when the loading is *eccentric*.



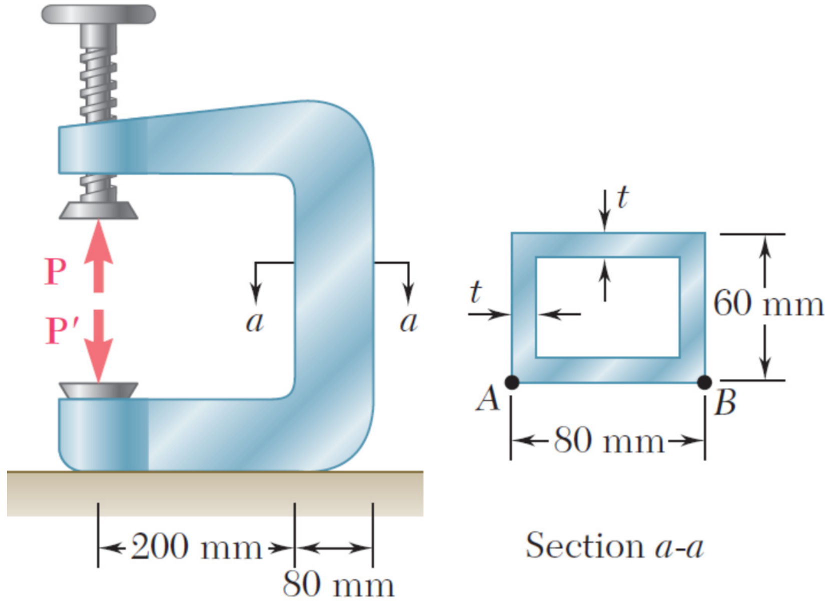


$$\sigma_x = \frac{P}{A} + \frac{My}{I}$$





Example 10: The vertical portion of the press shown consists of a rectangular tube of wall thickness t = 10 mm. Knowing that the press has been tightened on wooden planks being glued together until P = 20 kN, determine the stress at (a) point A, (b) point B.



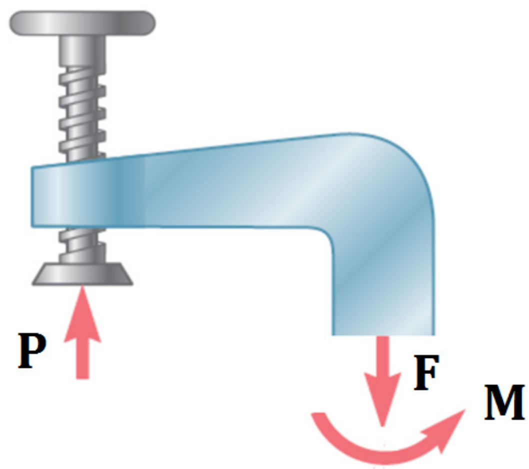
Calculating *P* and *M* at *a-a* section:

$$\sum F_y = 0 \rightarrow F = P = 20\,000\ N$$

$$\sum M = 0 \rightarrow M = P(240\ mm)$$

$$= 20\,000\ N \times 240\ mm$$

$$= 4800\,000\ Nmm$$



Calculating section properties:

$$A = 60 \times 80 - 40 \times 60 = 2400\ mm^2$$

$$I_{N.A.} = \frac{1}{12}60 \times 80^3 - \frac{1}{12}40 \times 60^3$$

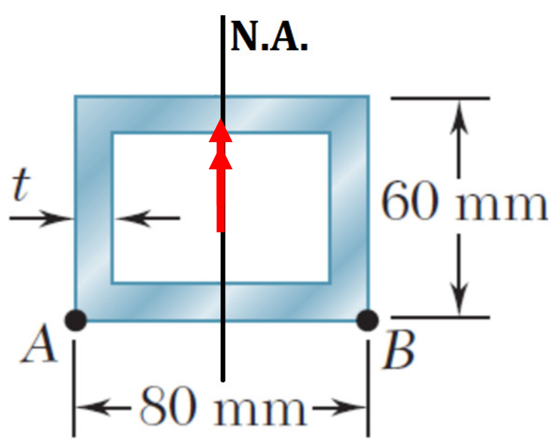
$$= 1\,840\,000\ mm^4$$

Stress at point A:

$$\sigma = \frac{P}{A} + \frac{Mc}{I}$$

$$= \frac{20\,000\ N}{2400\ mm^2} + \frac{4800\,000\ Nmm \times 40\ mm}{1\,840\,000\ mm^4}$$

$$= 8.33\ MPa + 104.35\ MPa = 112.7\ MPa$$

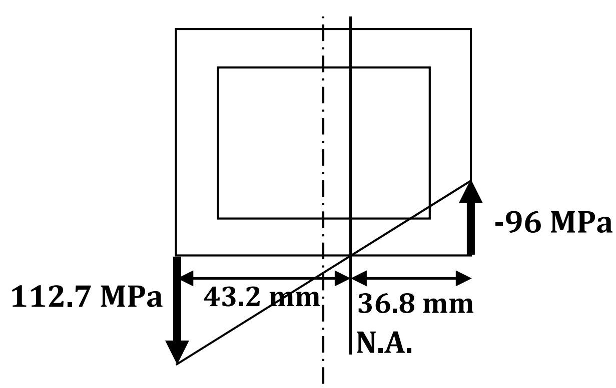


Stress at point B:

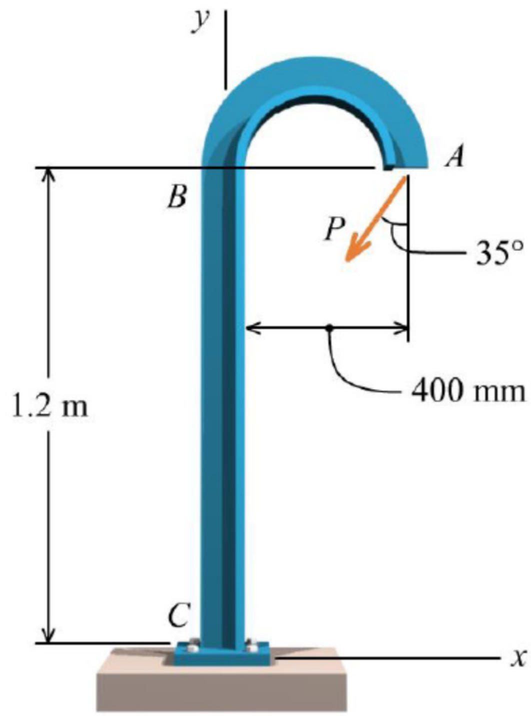
$$\sigma = \frac{P}{A} - \frac{Mc}{I}$$

$$= \frac{20\,000\ N}{2400\ mm^2} - \frac{4800\,000\ Nmm \times 40\ mm}{1\,840\,000\ mm^4}$$

$$= 8.33\ MPa - 104.35\ MPa = -96\ MPa$$



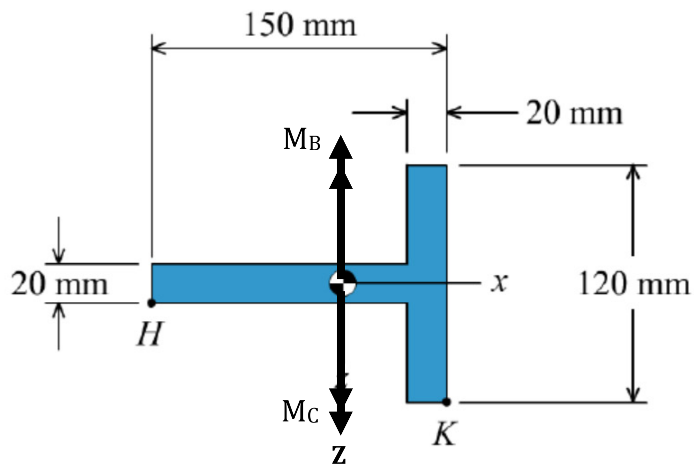
TBR 4: Determine the magnitudes and locations of the maximum tension and compression normal stresses within the vertical portion *BC* of the post (*P* = 25 kN).



Answer:

Maximal tensile stress: 82.2 MPa at section B

Maximal compressive stress: -79.3 MPa at section C



$\bar{x} = 101\ mm\ from\ lefside\ to\ centroid, I_z = 10\ 761\ 666.67\ mm^4$

Internal forces and moments

$F = (25\text{ kN})\cos 35° = 20.4788\text{ kN} = 20,478.8\text{ N}$ (vertical component)

$V = (25\text{ kN})\sin 35° = 14.3394\text{ kN} = 14,339.4\text{ N}$ (horizontal component)

at B $M_z = -(20,478.8\text{ N})(400\text{ mm} + 49.0\text{ mm}) = -9,194,981.2\text{ N-mm}$

at C $M_z = -(20,478.8\text{ N})(400\text{ mm} + 49.0\text{ mm}) + (14,339.4\text{ N})(1,200\text{ mm}) = 8,012,298.8\text{ N-mm}$

Normal stress at *H* at location *B*

$$\sigma_{\text{axial}} = \frac{F}{A} = \frac{-20,478.8\text{ N}}{5,000\text{ mm}^2} = -4.0958\text{ MPa}$$

$$\sigma_{H,\text{bending}} = \frac{M_z x}{I_z} = \frac{(\ 9,194,981.2\text{ N-mm})(\ 101.0\text{ mm})}{10,761,666.67\text{ mm}^4} = 86.2964\text{ MPa}$$

$$\sigma_H = -4.0958\text{ MPa} + 86.2964\text{ MPa} = 82.2\text{ MPa}$$

Normal stress at *H* at location *C*

$$\sigma_{H,\text{bending}} = \frac{M_z x}{I_z} = \frac{-(8,012,298.8\text{ N-mm})(\ 101.0\text{ mm})}{10,761,666.67\text{ mm}^4} = -75.1967\text{ MPa}$$

$$\sigma_H = -4.0958\text{ MPa} - 75.1967\text{ MPa} = -79.3\text{ MPa}$$

Normal stress at *K* at location *B*

$$\sigma_{K,\text{bending}} = \frac{M_z x}{I_z} = \frac{-(\ 9,194,981.2\text{ N-mm})(49.0\text{ mm})}{10,761,666.67\text{ mm}^4} = -41.8666\text{ MPa}$$

$$\sigma_K = -4.0958\text{ MPa} - 41.8666\text{ MPa} = -46.0\text{ MPa}$$

Normal stress at *K* at location *C*

$$\sigma_{K,\text{bending}} = \frac{M_z x}{I_z} = \frac{(8,012,298.8\text{ N-mm})(49.0\text{ mm})}{10,761,666.67\text{ mm}^4} = 36.4816\text{ MPa}$$

$$\sigma_K = -4.0958\text{ MPa} + 36.4816\text{ MPa} = 32.4\text{ MPa}$$

Maximum tension stress

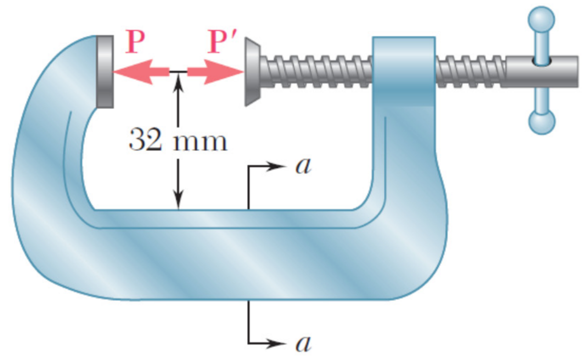
$\sigma_{\text{max tension}} = \boxed{82.2\text{ MPa (T)}}$ at location *B*

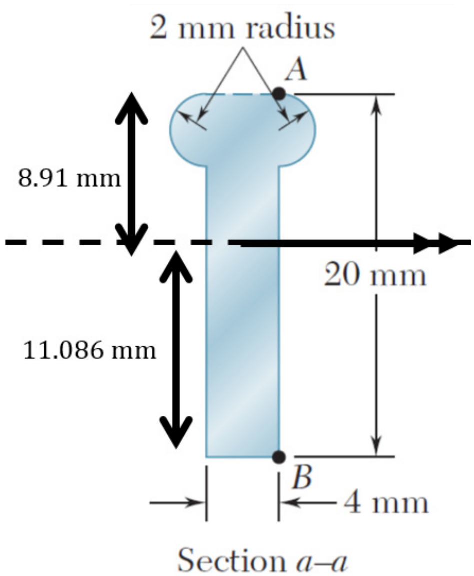
Maximum compression stress

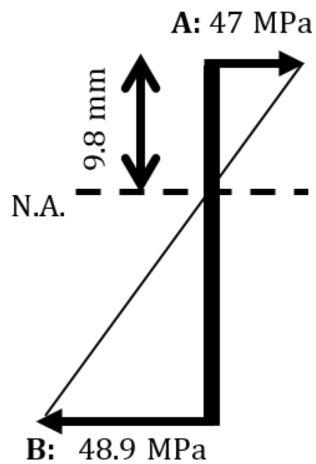
$\sigma_{\text{max compression}} = \boxed{79.3\text{ MPa (C)}}$ at location *C*

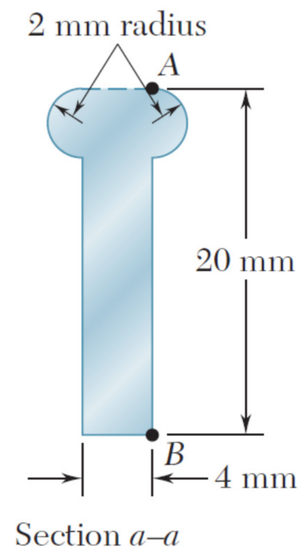
(83)

TBR 5: Maximal tensile and compressive stresses at *a-a* are equal to 47 and 67 MPa, respectively. Find maximal allowable value of P. Based on the calculated P find the position of neutral axis (1392).









$$\bar{y} = \frac{10 \times (4 \times 20) + 18 \times (\pi \times 2^2)}{4 \times 20 + \pi \times 2^2} = 11.086\, mm$$

$$I = \frac{1}{12} \times 4 \times 20^3 + 4 \times 20 \times (11.086 - 10)^2 + \frac{\pi}{4}(2^4) + \pi(2^2)(8.91 - 2)^2 = 3373.6\, mm^4$$

$$M = P \times (32\, mm + 8.91\, mm) = 40.91\, P$$

$$\sigma_{tension\_MAX} = \sigma_A = \frac{P}{A} + \frac{MC_A}{I} = \frac{P}{4 \times 20 + \pi \times 2^2} + \frac{(40.91\, P)(8.91\, mm)}{3373.6\, mm^4} = 47\, MPa \rightarrow P = 395.5\, N$$

$$\sigma_{comp\_MAX} = \sigma_B = \frac{P}{A} - \frac{MC_B}{I} = \frac{P}{4 \times 20 + \pi \times 2^2} - \frac{(40.91\, P)(11.086\, mm)}{3373.6\, mm^4} = -67\, MPa \rightarrow P = 541.9\, N$$

$$\rightarrow P_{max} = 395.5\, N$$

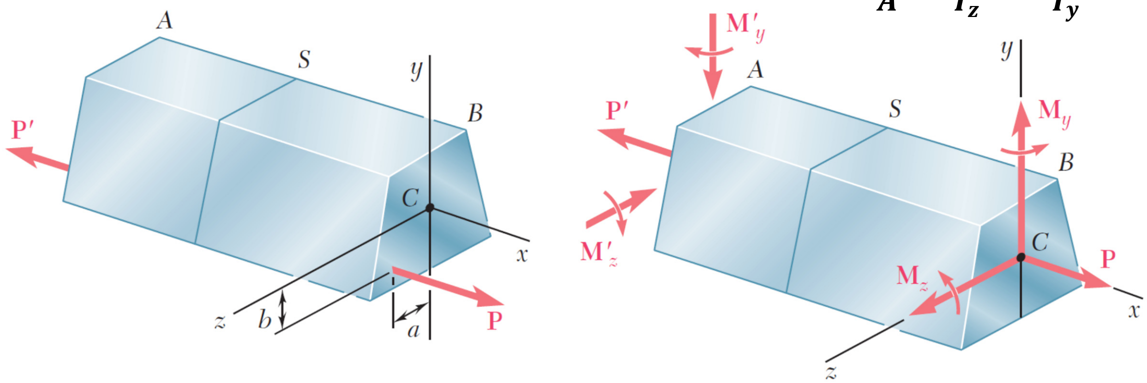
Neutral Axis:

$$\sigma_A = \frac{P}{A} + \frac{MC_A}{I} = \frac{395.5}{4 \times 20 + \pi \times 2^2} + \frac{(40.91 \times 395.5\, Nmm)(8.91\, mm)}{3373.6\, mm^4} = 47\, MPa$$

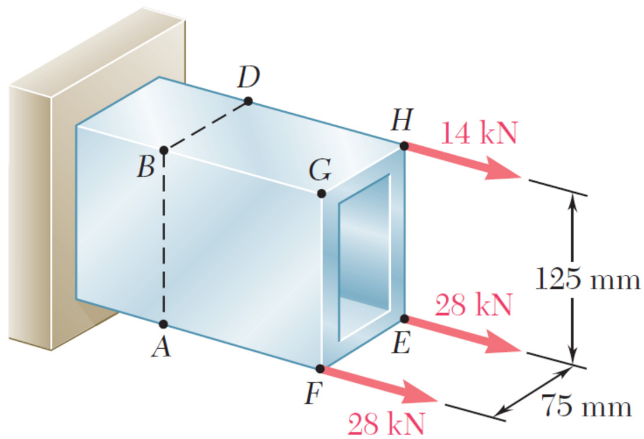
$$\sigma_B = \frac{P}{A} - \frac{MC_B}{I} = \frac{395.5}{4 \times 20 + \pi \times 2^2} - \frac{(40.91 \times 395.5\, Nmm)(11.086\, mm)}{3373.6\, mm^4} = -48.9\, MPa$$

General Case of Eccentric Axial Loading

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$



Example 11: The tube shown has a uniform wall thickness of 12 mm. For the loading given, determine (a) the stress at points A and B, (b) the point where the neutral axis intersects line ABD.



$A = 125 \times 75 - (125-24)(75-24) = 4224\, mm^2$

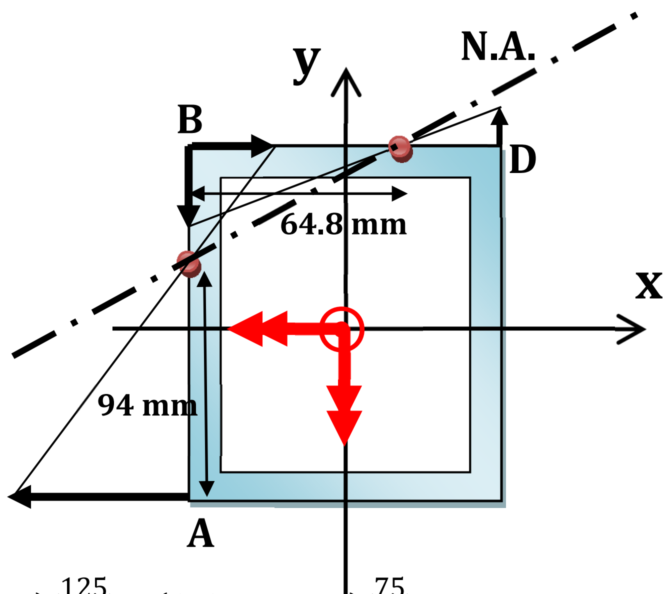
$$M_x = 14\, kN \left(\frac{125}{2}\, mm\right) - 2 \times 28\, kN \left(\frac{125}{2}\right)$$

$$M_y = 28\, kN \left(\frac{75}{2}\right) - (14+28)\, kN \left(\frac{75}{2}\, mm\right)$$

$$M_x = -2625\, kNmm,\ M_y = -525\, kNmm$$

$$I_x = \frac{1}{12} 75 \times 125^3 - \frac{1}{12}(75-24)(125-24)^3 = 7\,828\,252\, mm^4$$

$$I_y = \frac{1}{12} 125 \times 75^3 - \frac{1}{12}(125-24)(75-24)^3 = 3\,278\,052\, mm^4$$



$$\sigma_A = \frac{P}{A} + \frac{M_x y_A}{I_x} + \frac{M_y x_A}{I_y} = \frac{70\,000\, N}{4224\, mm^2} + \frac{(2\,625\,000\, Nmm)(\frac{125}{2})}{7\,828\,252\, mm^4} - \frac{(525\,000\, Nmm)(\frac{75}{2})}{3\,278\,052\, mm^4} = 31.52\, MPa$$

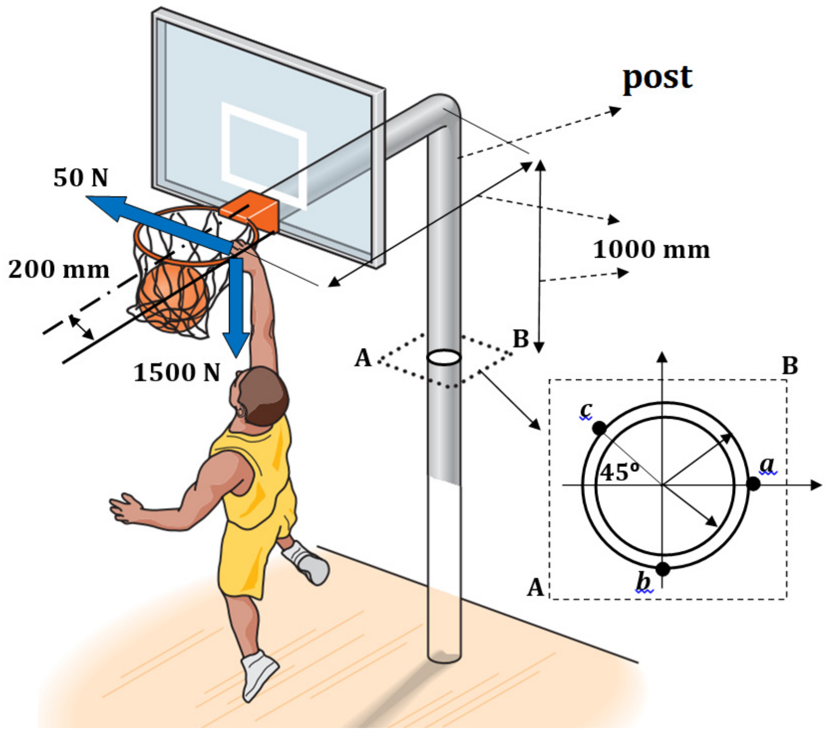
$$\sigma_B = \frac{P}{A} + \frac{M_x y_B}{I_x} + \frac{M_y x_B}{I_y} = \frac{70\,000\, N}{4224\, mm^2} - \frac{(2\,625\,000\, Nmm)(\frac{125}{2})}{7\,828\,252\, mm^4} - \frac{(525\,000\, Nmm)(\frac{75}{2})}{3\,278\,052\, mm^4} = -10.39\, MPa$$

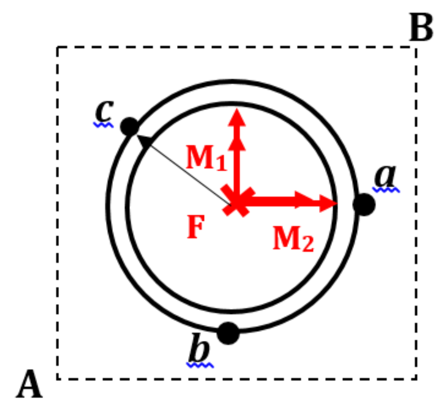
$$\sigma_D = \frac{P}{A} + \frac{M_x y_D}{I_x} + \frac{M_y x_D}{I_y} = \frac{70\,000\, N}{4224\, mm^2} - \frac{(2\,625\,000\, Nmm)(\frac{125}{2})}{7\,828\,252\, mm^4} + \frac{(525\,000\, Nmm)(\frac{75}{2})}{3\,278\,052\, mm^4} = 1.62\, MPa$$

Alternatively to find N.A.: $\frac{70\,000\, N}{4224\, mm^2} - \frac{(2\,625\,000\, Nmm)(y)}{7\,828\,252\, mm^4} + \frac{(525\,000\, Nmm)(x)}{3\,278\,052\, mm^4} = 0 \rightarrow$

$$16.57 - 0.335y + 0.16\, x = 0\ (line\ equation\ of\ N.A.)$$

TBR 6: The basketball player applies the forces shown to the basket ring. The post has a circular cross section with internal and external radius of 150 and 200 mm. Find stress at points *a*, *b*, and *c* on the outer surface of the post at section AB (1393).





$F = 1500\ N\ (compressive)$ 1

$M_1 = 1500\ N\ \times 200\ mm - 50\ N\ \times 1000\ mm = 250\ 000\ Nmm$ 3

$M_2 = 1500\ N\ \times 1000\ mm = 1500\ 000\ Nmm$ 3

$$\sigma_a = \frac{-F}{A} - \frac{M_1 r_a}{I} = \frac{-1500\ N}{\pi(200^2 - 150^2)mm^2} - \frac{(250\ 000\ Nmm)(200\ mm)}{\frac{\pi}{4}(200^4 - 150^4)}$$ 5

$\sigma_a = -0.085\ MPa = \ -85\ kPa$ 1

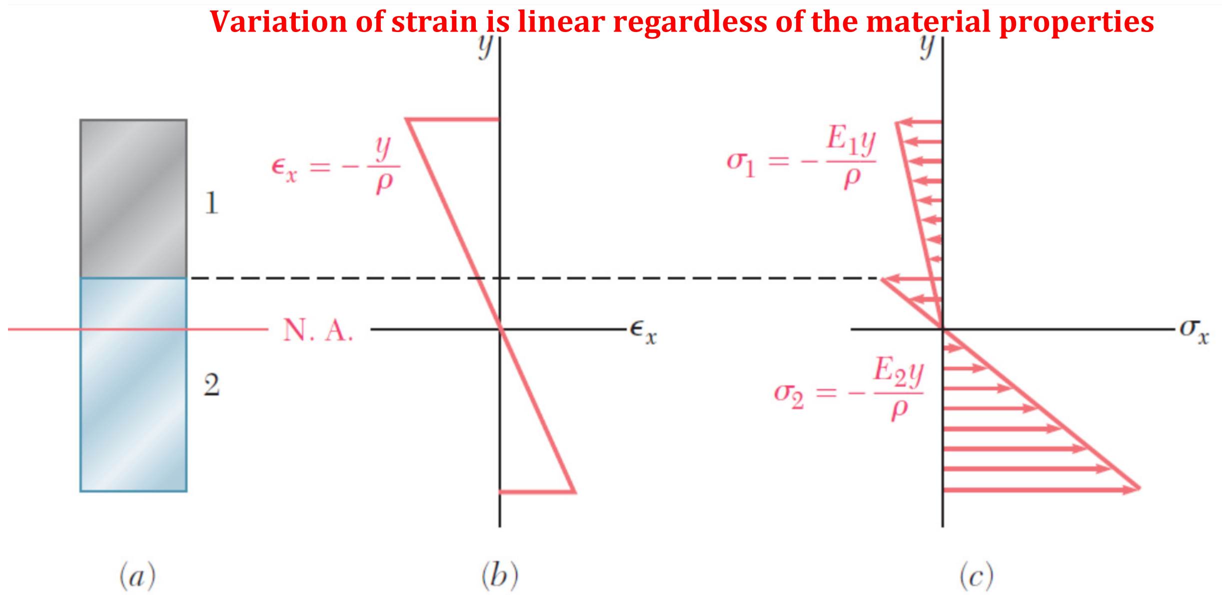
$$\sigma_b = \frac{-F}{A} - \frac{M_2 r_b}{I} = \frac{-1500\ N}{\pi(200^2 - 150^2)mm^2} - \frac{(1500\ 000\ Nmm)(200\ mm)}{\frac{\pi}{4}(200^4 - 150^4)}$$ 5

$\sigma_b = -0.38\ MPa = \ -380\ kPa$ 1

$$\sigma_c = \frac{-F}{A} + \frac{M_1(200\cos 45°)}{I} + \frac{M_2(200\sin 45°)}{I}$$
$$= \frac{-1500\ N}{\pi(200^2 - 150^2)mm^2} + \frac{(250\ 000\ Nmm)(141.4\ mm)}{\frac{\pi}{4}(200^4 - 150^4)}$$
$$+ \frac{(1500\ 000\ Nmm)(141.4\ mm)}{\frac{\pi}{4}(200^4 - 150^4)}$$ 5

$\sigma_c = 0.26\ MPa = 260\ kPa$ 1

Bending of Members Made of Several Materials ($E_2 > E_1$)



Strain and stress distribution in bar made of two materials

To determine position of the neutral axis we convert one material to another so that:

$$1 - M_{max})_{before\ transformation} = M_{max})_{after\ transformation}$$

2- Strain distribution remains unchanged so to have the same N.A. for the transformed section

(for material 2 only assuming that material 1 does not exist)

So for material 2 alone the N. A. passes through its centroid. As material 1 is weaker we expect that:

The transformed section becomes bigger in order to have the same bending resistance

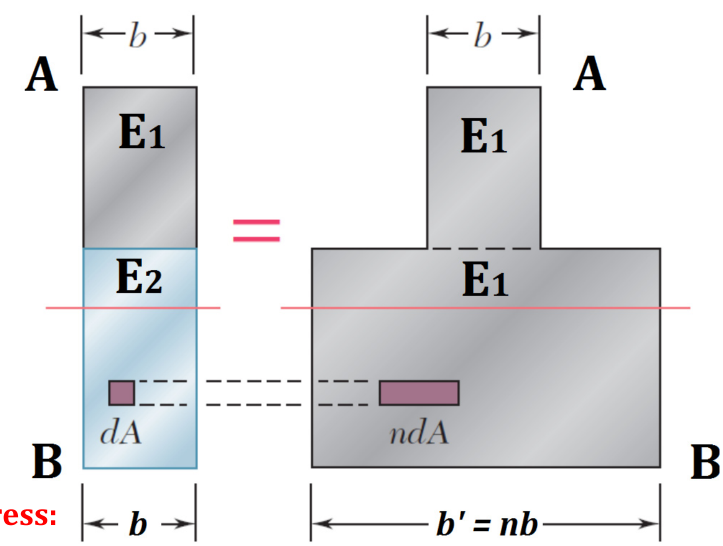
$$\frac{\sigma_{max})_i I_i}{C_B} = \frac{\sigma_{max})_f I_f}{C_B}$$

$$\rightarrow E_2 \varepsilon_{max})_i I_i = E_1 \varepsilon_{max})_f I_f$$

$$\rightarrow E_2 I_i = E_1 I_f \rightarrow I_f = \frac{E_2}{E_1} I_i = n I_i$$

For rectangular cross section:

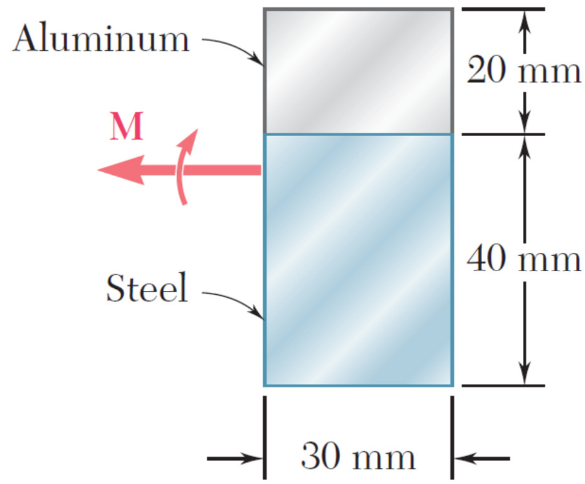
$$n = \frac{E_2}{E_1} = \frac{I_f}{I_i} = \frac{\frac{1}{12} b' h^3}{\frac{1}{12} b h^3} = \frac{b'}{b} \rightarrow b' = nb$$



As position of NA is determined we can calculate stress:

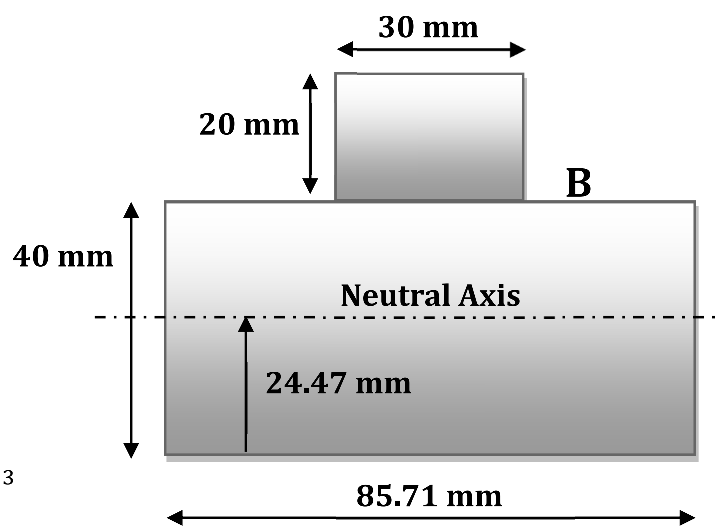
$$\sigma_A = E_1 \varepsilon_A)_i = E_1 \varepsilon_A)_f = E_1 \frac{\sigma_A)_f}{E_1} = \frac{M c_A}{I_{N.A.}}, \quad \sigma_B = E_2 \varepsilon_B)_i = E_2 \varepsilon_B)_f = E_2 \frac{\sigma_B)_f}{E_1} = \boldsymbol{n} \frac{M c_B}{I_{N.A.}}$$

Example 5: A steel bar and an aluminium bar are bonded together to form the composite beam shown. The modulus of elasticity for aluminum is 70 GPa and for steel is 200 GPa. Knowing that the beam is bent about a horizontal axis by a couple of moment M=1500 Nm, determine the maximum stress in (*a*) the aluminums, (*b*) the steel.



$$n = \frac{E_s}{E_a} = \frac{200\ GPa}{70\ GPa} = 2.857$$

New width = 30 × 2.857 = 85.71 mm



$$\bar{y} = \frac{20 \times (40 \times 85.71) + 50 \times (20 \times 30)}{40 \times 85.71 + 20 \times 30}$$
$$= 24.47\ mm$$

$$I_{N.A.} = \frac{1}{12} \times 85.71 \times 40^3 + 85.71 \times 40$$
$$\times (24.47 - 20)^2 + \frac{1}{12} \times 30 \times 20^3$$
$$+ 30 \times 20 \times (50 - 24.47)^2$$
$$= 936717.3\ mm^4$$

$$\sigma_{\max\,(AL)} = -\frac{1500 \times 10^3\ Nmm \times (60 - 24.47)\ mm}{936717.3\ mm^4} = -56.9\ MPa$$

$$\sigma_{\max\,(ST)} = \frac{(\mathbf{2.857}) \times 1500 \times 10^3\ Nmm \times (24.47)\ mm}{936717.3\ mm^4} = +111.9\ MPa$$

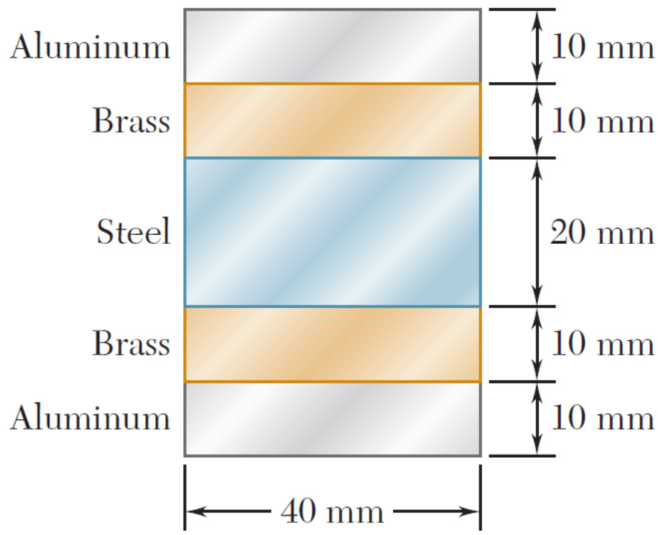
$$\frac{1}{\rho} = \frac{M}{EI} = \frac{M_{AL}}{E_{AL} I_{AL}} = \frac{1500\,000\ Nmm}{(70\,000\ MPa) \times\ 936717.3\ mm^4} = 2.287 \times 10^{-5} \frac{1}{mm} \rightarrow \rho = 43713.5\ mm$$
$$= 43.7\ m$$

Attention:

$$\sigma_{\mathrm{B}\ (ST)} = -\frac{(\mathbf{2.857}) \times 1500 \times 10^3\ Nmm \times (40 - 24.47)\ mm}{936717.3\ mm^4} = -71.1\ MPa$$

$$\sigma_{\mathrm{B}\ (AL)} = -\frac{1500 \times 10^3\ Nmm \times (40 - 24.47)\ mm}{936717.3\ mm^4} = -24.9\ MPa$$

Example 6: Five metal strips, each 40 mm wide, are bonded together to form the composite beam shown. The modulus of elasticity is 210 GPa for the steel, 105 GPa for the brass, and 70 GPa for the aluminium. Knowing that the beam is bent about a horizontal axis by a couple of moment 1800 Nm, determine (*a*) the maximum stress in each of the three metals, (*b*) the radius of curvature of the composite beam.

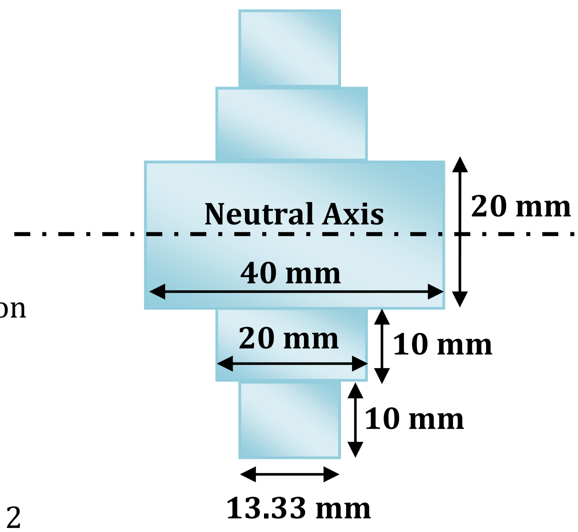


$$n_1 = \frac{E_b}{E_s} = \frac{105\ GPa}{210\ GPa} = \frac{1}{2}$$

$$n_2 = \frac{E_a}{E_s} = \frac{70\ GPa}{210\ GPa} = \frac{1}{3}$$

New width for Brass: 1/2 × 40 = 20 mm

New width for Aluminium: 1/3 × 40 = 13.33 mm



Neutral axis passes through the centroid of the section which is located at its middle due to the symmetry.

$$I_{N.A.} = \frac{1}{12} \times 40 \times 20^3 + 2 \times \left(\frac{1}{12} \times 20 \times 10^3 + 20 \times 10 \times 15^2\right) + 2 \times \left(\frac{1}{12} \times 13.33 \times 10^3 + 13.33 \times 10 \times 25^2\right) = 288888.9\ mm^4$$

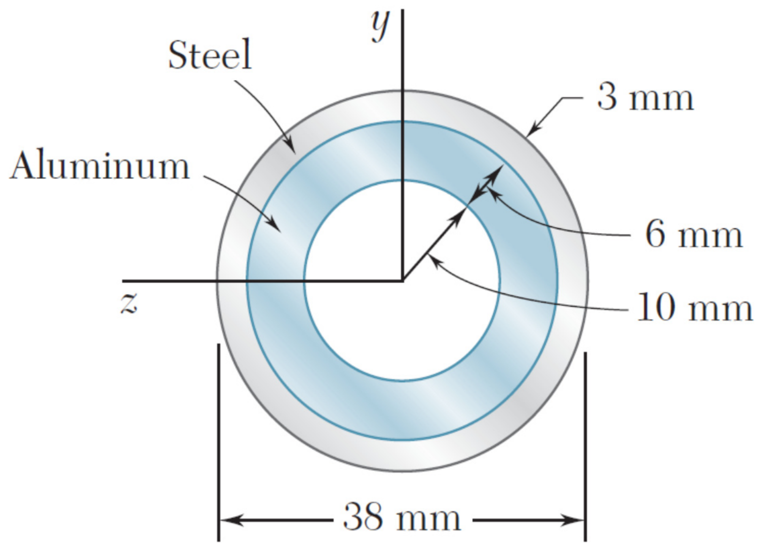
$$\sigma_{\max\,(ST)} = \pm \frac{1800 \times 10^3\ Nmm \times 10\ mm}{288888.9\ mm^4} = \pm 62.3\ MPa$$

$$\sigma_{\max\,(BR)} = \pm \frac{\mathbf{\frac{1}{2}} \times 1800 \times 10^3\ Nmm \times 20\ mm}{288888.9\ mm^4} = \pm 62.3\ MPa$$

$$\sigma_{\max\,(AL)} = \pm \frac{\mathbf{\frac{1}{3}} \times 1800 \times 10^3\ Nmm \times 30\ mm}{288888.9\ mm^4} = \pm 62.3\ MPa$$

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{1800\,000\ Nmm}{(210\,000\ MPa) \times 288888.9\ mm^4} = 2.967 \times 10^{-5}\,\frac{1}{mm} \rightarrow \rho = 33703\ mm = 33.7\ m$$

Example 7: A steel pipe and an aluminium pipe are securely bonded together to form the composite beam shown. The modulus of elasticity is 210 GPa for the steel and 70 GPa for the aluminium. Knowing that the composite beam is bent by a couple of moment 500 Nm, determine the maximum stress (a) in the aluminium, (b) in the steel.



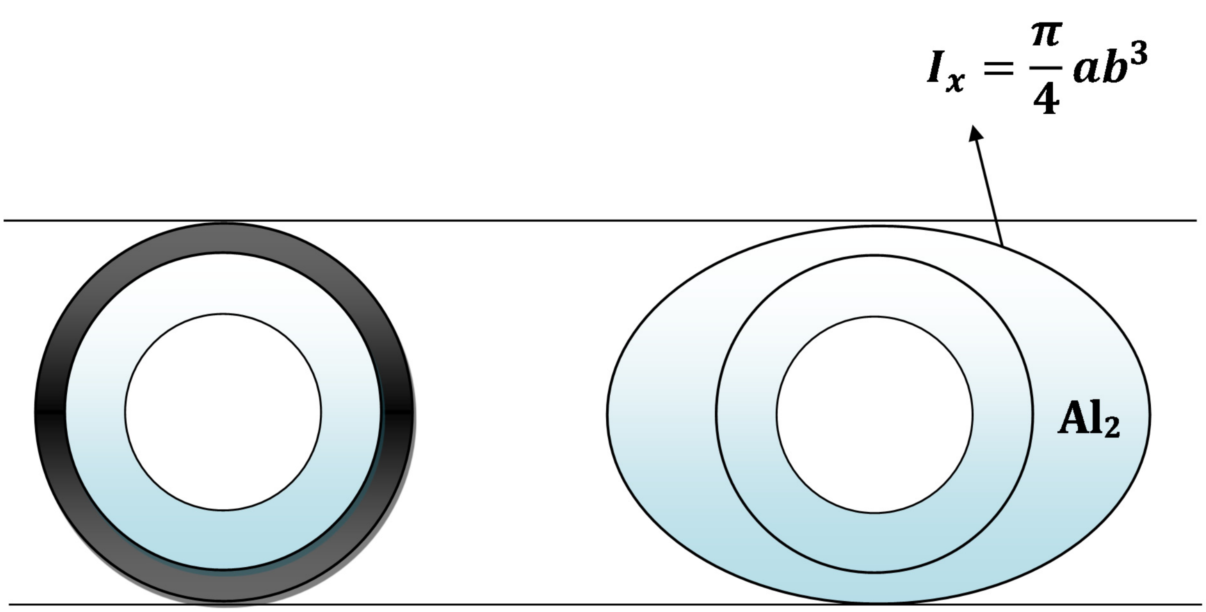
$$\frac{E_{st}}{E_{Al2}} = \frac{I_{Al2}}{I_{st}} \rightarrow 3 = \frac{I_{Al2}}{\frac{\pi}{4}(19^4 - 16^4)} \rightarrow I_{Al2} = 152.64 \times 10^3 mm^4$$

$$I_{Total} = I_{Al2} + \frac{\pi}{4}(16^4 - 10^4) = 196.26 \times 10^3\ mm^4$$

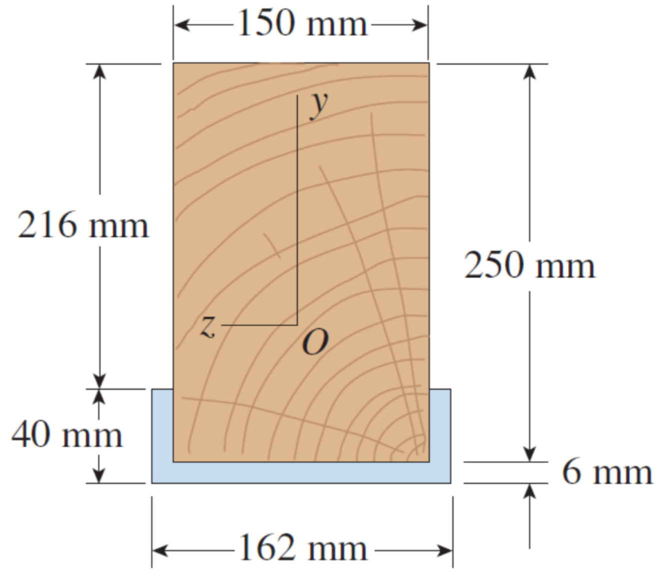
$$\sigma_{Al_{max}} = \frac{500\,000\ Nmm \times (\pm 16\ mm)}{196.26 \times 10^3\ mm^4} = \pm 40.8\ MPa$$

$$\sigma_{St_{max}} = \frac{3 \times 500\,000\ Nmm \times (\pm 19\ mm)}{196.26 \times 10^3\ mm^4} = \pm 145.2\ MPa$$

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{500\,000\ Nmm}{(70\,000\ MPa) \times 196.26 \times 10^3\ mm^4} = 3.639 \times 10^{-5} \frac{1}{mm} \rightarrow \rho = 27476mm$$
$$= 27.5\ m$$



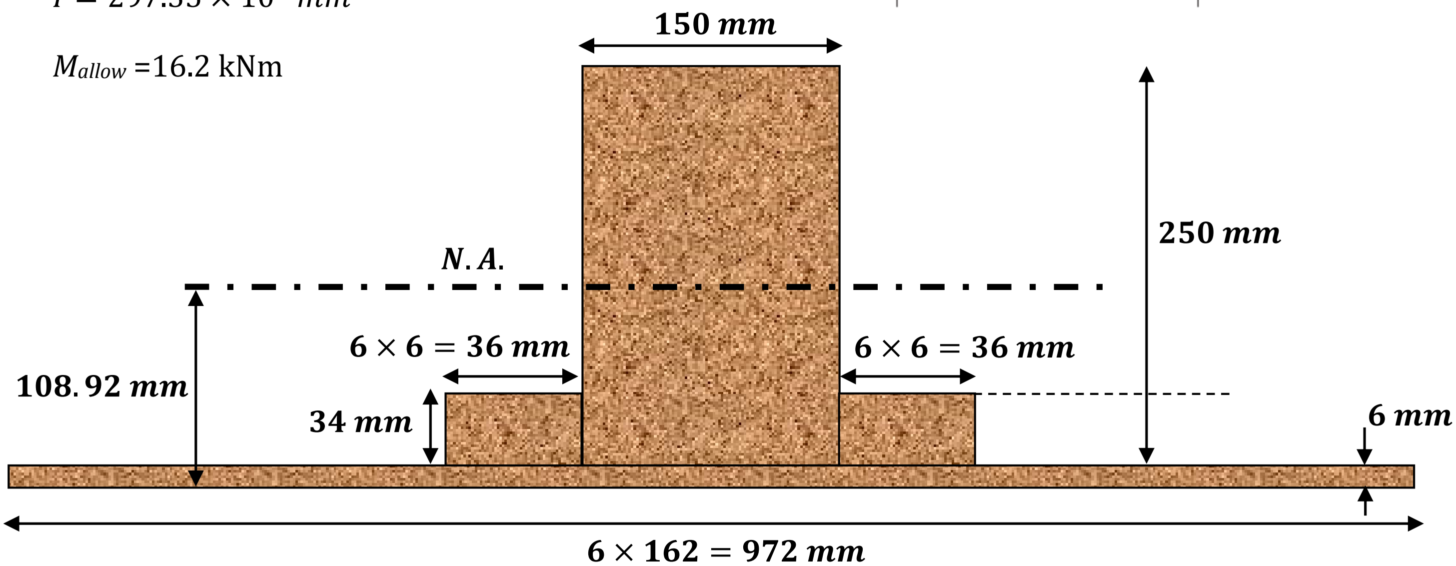
TBR 7: A wood beam reinforced by an aluminium channel section is shown in the figure. The beam has a cross section of dimensions 150 mm by 250 mm, and the channel has a uniform thickness of 6 mm. If the allowable stresses in the wood and aluminium are 8.0 MPa and 38 MPa, respectively, and if their modulus of elasticity are in the ratio 1 to 6, what is the maximum allowable bending moment for the beam?



Answer: $\bar{y} = 108.92\ mm\ from\ the\ base$

$I = 297.35 \times 10^6\ mm^4$

M_{allow} =16.2 kNm



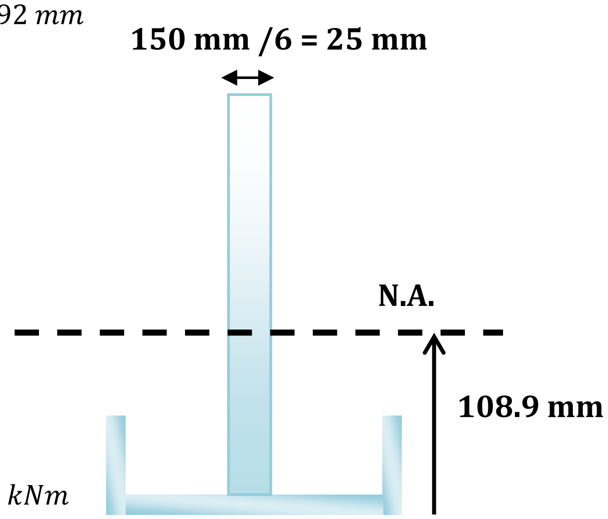
$\boldsymbol{Transformed\ section}$ (*wood*)

$$\bar{y} = \frac{131 \times (25 \times 250) + 2 \times \{20 \times (40 \times 6)\} + 3 \times (150 \times 6)}{25 \times 250 + 2 \times 40 \times 6 + 150 \times 6} = 108.92\ mm$$

$$I_{N.A.} = \frac{1}{12} \times 25 \times 250^3 + 25 \times 250 \times (131 - 108.92)^2 + 2 \times \left(\frac{1}{12} \times 6 \times 40^3 + 6 \times 40 \times (108.92 - 20)^2\right) + \left(\frac{1}{12} \times 150 \times 6^3 + 150 \times 6 \times (108.92 - 3)^2\right) = 49558213\ mm^4$$

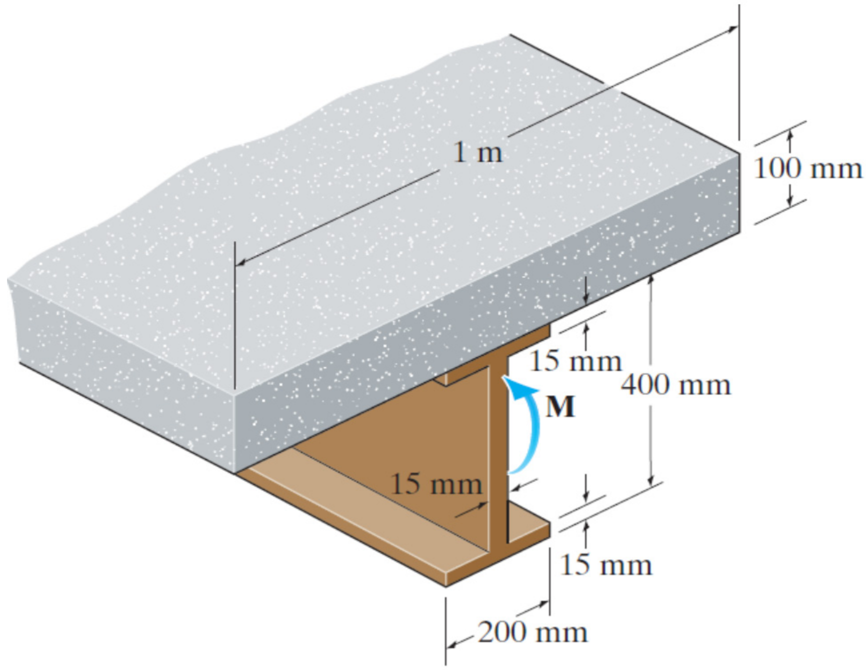
$$\sigma_{\max\,(AL)} = \frac{M \times 108.92\ mm}{49558213\ mm^4} = 38\ MPa \rightarrow M = 17.3\ kNm$$

$$\sigma_{\max\,(Wood)} = -\frac{1}{6} \times \frac{M \times (256 - 108.92)mm}{49558213\ mm^4} = -8\ MPa \rightarrow M = 16.2\ kNm$$



$\boldsymbol{Transformed\ section}$ (**Aluminium**)

TBR 8: The low strength concrete floor slab (σ_Y = 10 MPa, E = 22.1 GPa) is integrated with a wide-flange A-36 steel beam (σ_Y = 165 MPa, E = 200 GPa) using shear studs (not shown) to form the composite beam. If the allowable bending stress for the concrete is and allowable bending stress for steel is determine the maximum allowable internal moment M that can be applied to the beam. Also find the curvature based on the calculated maximal moment (1390).



Answer: $\boldsymbol{M_{allow}}$ **=330 kNm**

Section Properties: The beam cross section will be transformed into that of steel. Here, $n = \dfrac{E_{con}}{E_{st}} = \dfrac{22.1}{200} = 0.1105$. Thus, $b_{st} = nb_{con} = 0.1105(1) = 0.1105$ m. The location of the transformed section is

$$\bar{y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{0.0075(0.015)(0.2) + 0.2(0.37)(0.015) + 0.3925(0.015)(0.2) + 0.45(0.1)(0.1105)}{0.015(0.2) + 0.37(0.015) + 0.015(0.2) + 0.1(0.1105)}$$

$$= 0.3222 \text{ m}$$

The moment of inertia of the transformed section about the neutral axis is

$$I = \Sigma\bar{I} + Ad^2 = \frac{1}{12}(0.2)\left(0.015^3\right) + 0.2(0.015)(0.3222 - 0.0075)^2 + \frac{1}{12}(0.015)\left(0.37^3\right) + 0.015(0.37)(0.3222 - 0.2)^2$$

$$+ \frac{1}{12}(0.2)\left(0.015^3\right) + 0.2(0.015)(0.3925 - 0.3222)^2 + \frac{1}{12}(0.1105)\left(0.1^3\right) + 0.1105(0.1)(0.45 - 0.3222)^2 = 647.93\left(10^{-6}\right) \text{ m}^4$$

Bending Stress: Assuming failure of steel,

$$(\sigma_{\text{allow}})_{st} = \frac{Mc_{st}}{I}; \quad 165\left(10^6\right) = \frac{M(0.3222)}{647.93\left(10^{-6}\right)}$$

$$M = 331\,770.52 \text{ N}\cdot\text{m} = 332 \text{ kN}\cdot\text{m}$$

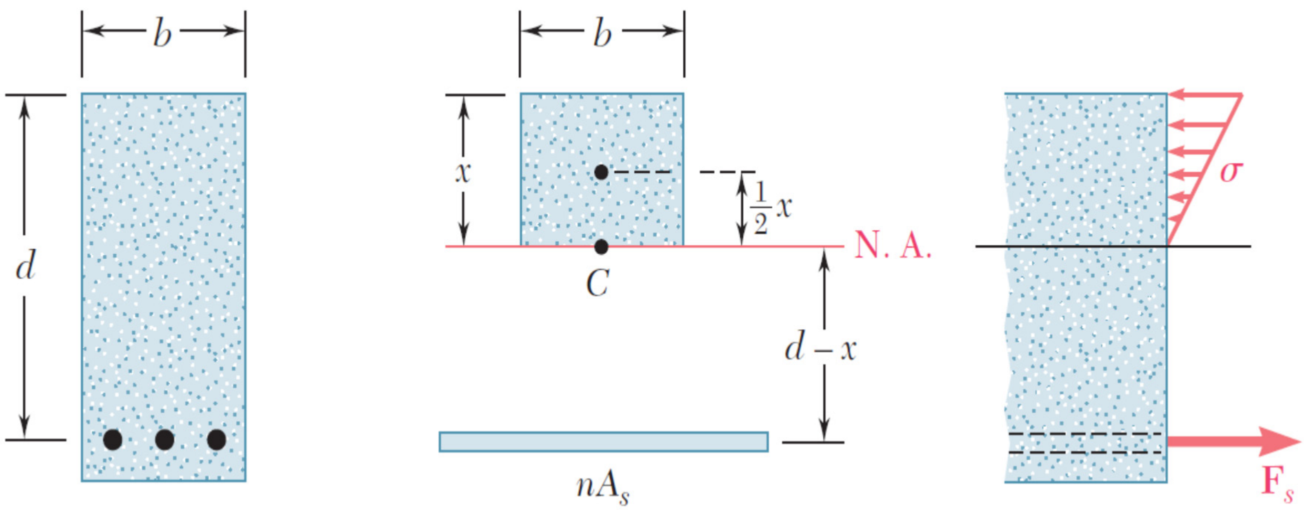
Assuming failure of concrete,

$$(\sigma_{\text{allow}})_{con} = n\frac{Mc_{con}}{I}; \qquad 10\left(10^6\right) = 0.1105\left[\frac{M(0.5 - 0.3222)}{647.93\left(10^{-6}\right)}\right]$$

$$M = 329\,849.77 \text{ N}\cdot\text{m} = 330 \text{ kN}\cdot\text{m (controls)} \quad \textbf{Ans.}$$

Reinforced Concrete Beams

An important example of structural members made of two different materials is furnished by *reinforced concrete beams.* These beams, when subjected to positive bending moments, are reinforced by steel rods placed a short distance above their lower face. Fortunately, there is a natural bond between concrete and steel, so that no slipping occurs between them during bending. Since concrete is very weak in tension, it will crack below the neutral surface and the steel rods will carry the entire tensile load, while the upper part of the concrete beam will carry the compressive load. To be most effective, these rods are located farthest from the beam's neutral axis so that they resist the greatest possible tensile moment. The diameters of the rods are small compared to the depth of the cross section.



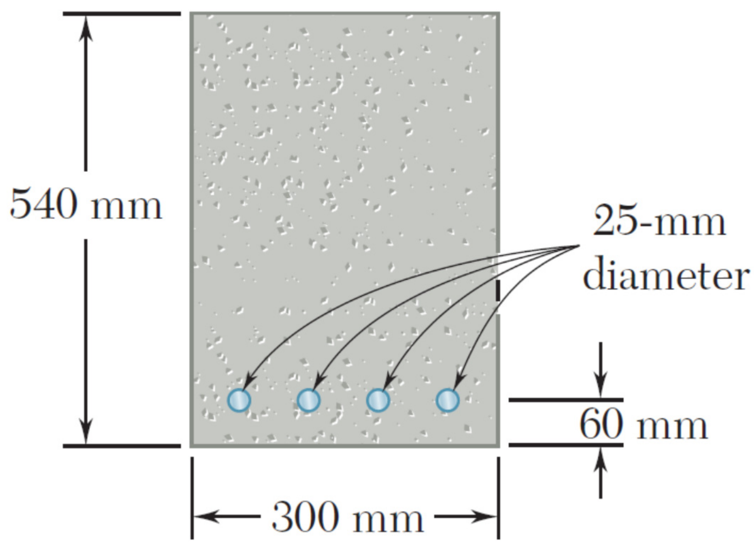
The position of the neutral axis is obtained by determining the distance x from the upper face of the beam to the centroid C of the transformed section. Denoting by *b* the width of the beam, and by *d* the distance from the upper face to the center line of the steel rods, we write that the first moment of the transformed section with respect to the neutral axis must be zero. Since the first moment of each of the two portions of the transformed section is obtained by multiplying its area by the distance of its own centroid from the neutral axis, we have:

$$bx\left(\frac{x}{2}\right) - nA_s\,(d-x) = 0 \rightarrow \frac{1}{2}bx^2 + nA_sx - nA_sd = 0$$

Solving this quadratic equation for x, we obtain both the position of the neutral axis in the beam, and the portion of the cross section of the concrete beam that is effectively used. The determination of the stresses in the transformed section is carried out as explained before. The distribution of the compressive stresses in the concrete and the resultant F_s of the tensile forces in the steel rods are shown.

$$\sigma_c = -\frac{Mx}{I_{N.A.}}, \qquad \sigma_s = n\frac{M(d-x)}{I_{N.A.}}$$

Example 9: The reinforced concrete beam shown is subjected to a positive bending moment of 175 kNm. Knowing that the modulus of elasticity is 25 GPa for the concrete and 200 GPa for the steel, determine (*a*) the stress in the steel, (*b*) the maximum stress in the concrete.



$$n = \frac{E_s}{E_c} = \frac{200\ GPa}{25\ GPa} = 8$$

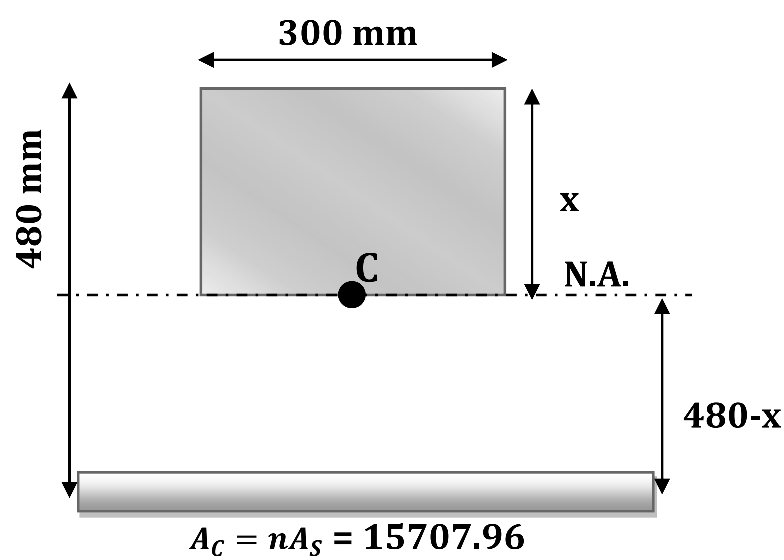
$$A_C = nA_S = 8 \times \left(4 \times \frac{\pi}{4} 25^2\right)$$
$$= 15707.96\ mm^2$$

Finding the neutral axis:

$$300x\frac{x}{2} - 15707.96(480 - x) = 0.$$

$$x^2 + 104.71x - 50\,265.472 = 0.$$

$$\boldsymbol{x = 177.87\ mm} \quad and\ x = -282.6\ mm$$



Transformed section (concrete)

Calculating the moment of inertia:

$$I_{N.A.} = \frac{1}{3}(300)(177.87)^3 + 15707.96$$
$$\times (480 - 177.87)^2$$
$$= 1.996 \times 10^9\ mm^4$$

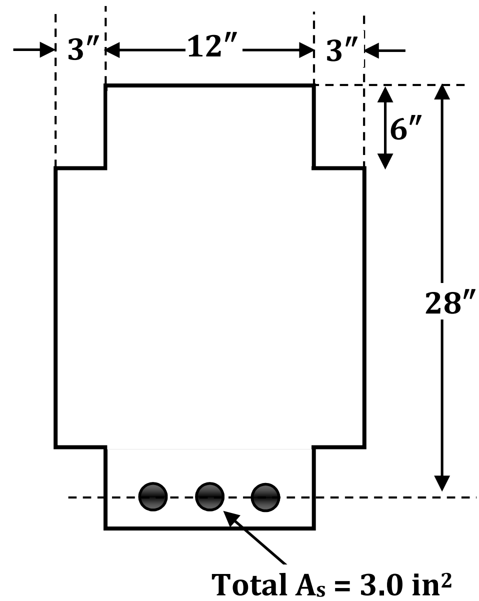
Stress in steel members:

$$\sigma_s = \frac{n \times Mc}{I} = \frac{\mathbf{8} \times 175\,000\,000\ Nmm \times (480 - 177.87)\ mm}{1.996 \times 10^9\ mm^4} = 211.9\ MPa$$

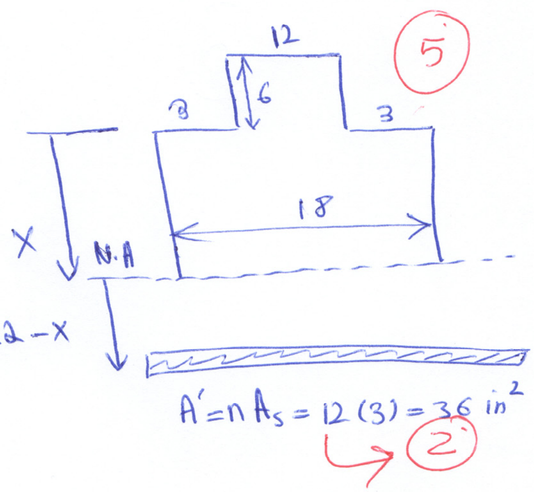
Stress in concrete member is compressive and is equal to:

$$\sigma_c = \frac{Mc}{I} = -\frac{175\,000\,000\ Nmm \times 177.87\ mm}{1.996 \times 10^9\ mm^4} = -15.59\ MPa$$

TBR 9: A beam has the cross section shown in figure, and is subject to a positive bending moment that causes a tensile stress in the steel of 20 ksi (20000 psi = 20000 lb/in^2). If n = 12 (elastic modulus of steel is 12 times greater than that of concrete) calculate the bending moment applied to the beam (1391).



$n = 12$



$\Sigma \bar{y}A = 0.$ 1

$(3+x)(6\times12) + \frac{x}{2}(18x) -$

$36(22-x) = 0.$ 4

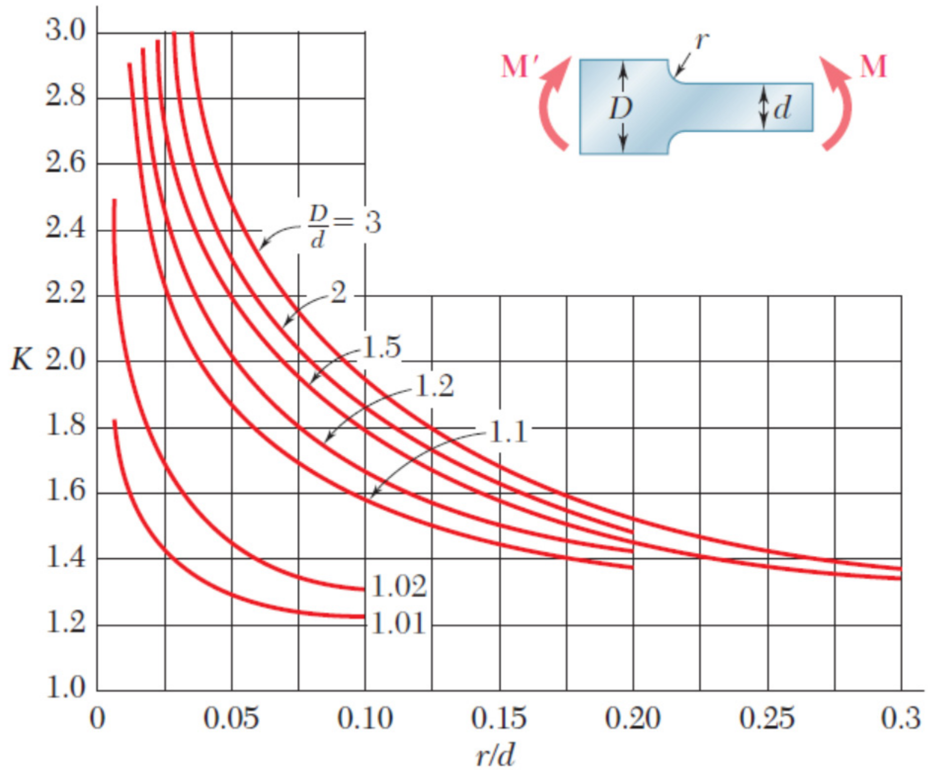
$\rightarrow x = 4 \text{ in}$

$$I = \frac{1}{12}12(6^3) + 12\times6\times(3+4)^2 +$$
$$\frac{1}{12}18(4^3) + (18\times4)(2)^2 + 36\times18^2$$
$$= 15.8\times10^3 \text{ in}^4$$ 5

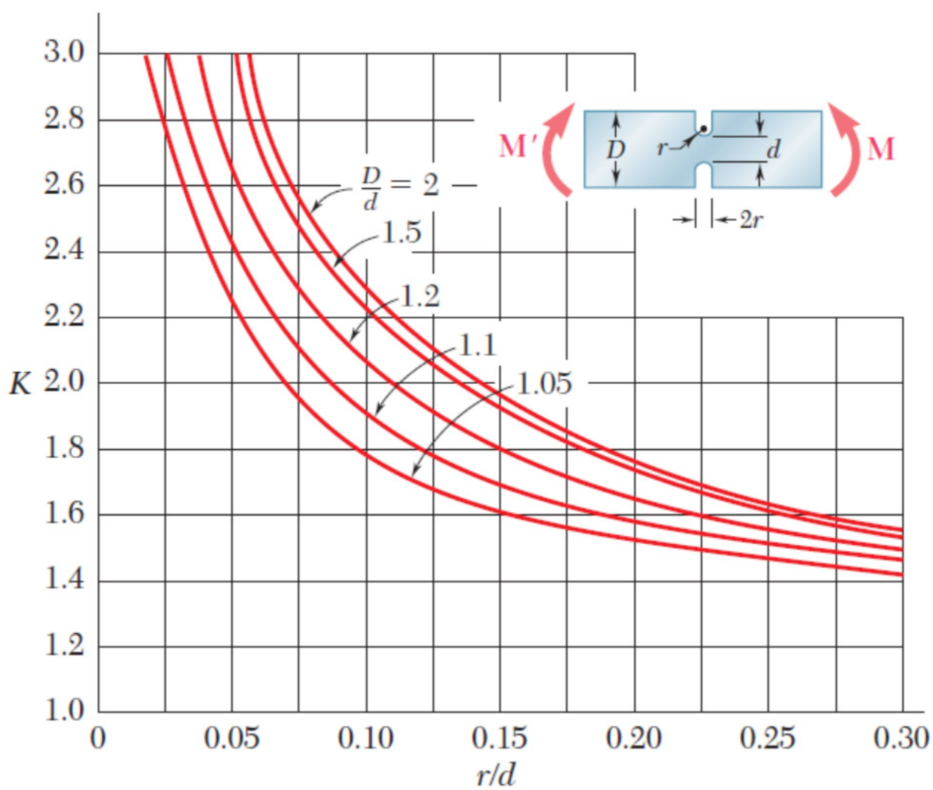
$\sigma_s = n\frac{Mc}{I} \rightarrow$ 2 $20\,000 \frac{\text{lb}}{\text{in}^2} = 12\frac{M(18 \text{ in})}{15.8\times10^3 \text{ in}^4}$

$\rightarrow M = 1.46\times10^6 \text{ lb.in}$ 1

STRESS CONCENTRATIONS



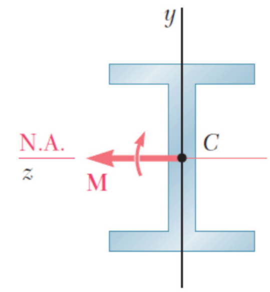
$$K = \frac{\sigma_{max}}{\frac{M\frac{d}{2}}{I}}$$



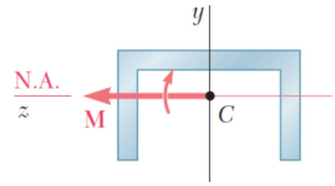
Review cross sectional properties from Statics: moment of inertia (Ix, Iy) and product of inertia for an area (Ixy)! Mohr's circle to determine principal axes of an area!

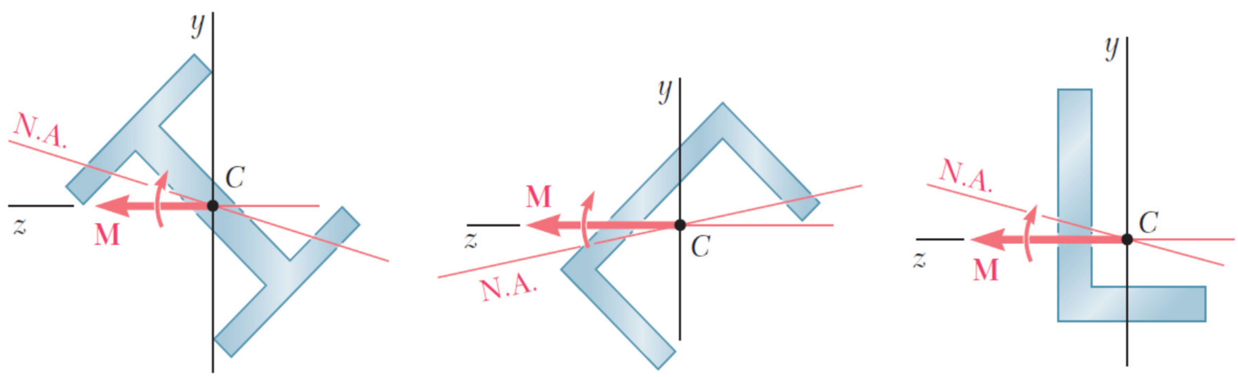
UNSYMMETRIC BENDING

Our analysis of pure bending has been limited so far to members possessing at least one plane of symmetry and subjected to couples acting in that plane. We found that the neutral axis of the cross section in symmetric bending passes through centroid of the section and coincides with the axis of the couple. Now consider situations where the bending couples do *not* act in a plane of symmetry of the member, either because they act in a different plane, or because the member does not possess any plane of symmetry. In such situations, we cannot assume that the member will bend in the plane of the couples. As shown, the couple exerted on the section has again been assumed to act in a vertical plane and has been represented by a horizontal couple vector **M.** However, since the vertical plane is not a plane of symmetry, *we cannot expect the member to bend in that plane, or the neutral axis of the section to coincide with the axis of the couple*.



(a)





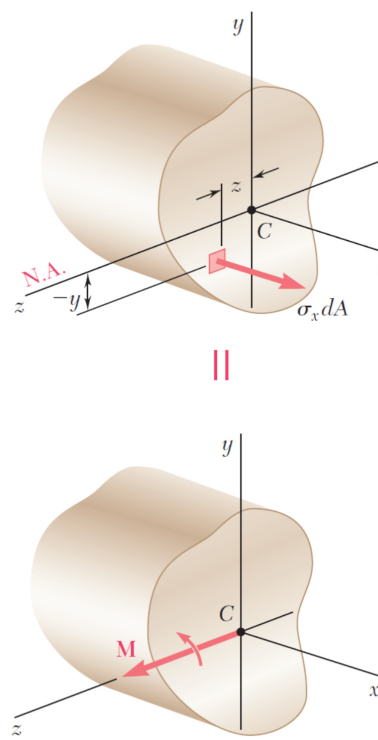
We assume that N.A. is directed toward an arbitrary z-axis. An arbitrary directed moment has a component toward z and a component toward y. We initially only consider M toward z.

$$\sum F_x = 0 \rightarrow \int \sigma_x \, dA = 0 \rightarrow N.A. \, passes \; through \; centroid$$

$$\sum M_z = 0 \rightarrow \int (-y\sigma_x \, dA) = M \rightarrow \sigma_x = -\frac{My}{I}$$

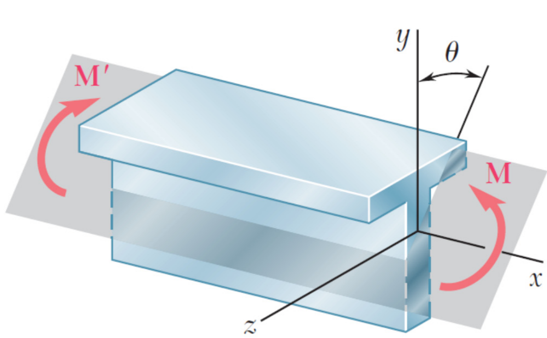
$$\sum M_y = 0 \rightarrow \int z\sigma_x \, dA = 0 \rightarrow \int z(-\frac{\sigma_{max} y}{c}) \, dA = 0 \rightarrow$$

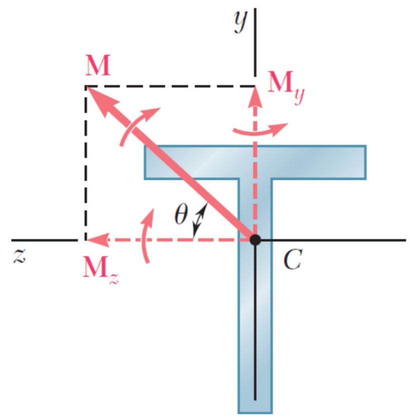
$$\int yz \, dA = 0 \rightarrow I_{yz} = 0 \rightarrow$$ ***y and z must be principal axes of the cross section***

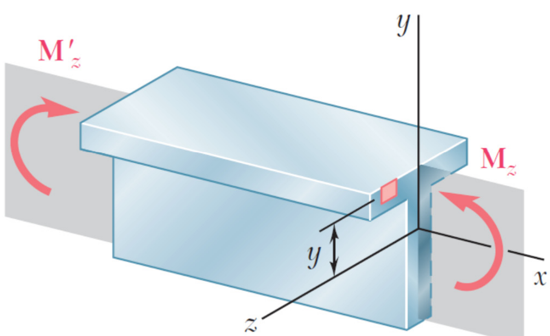


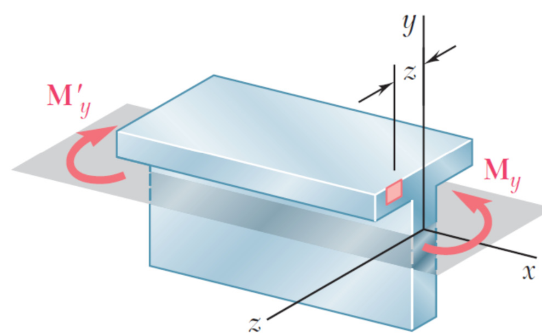
The first equation indicates that the N.A. passes through the centroid of the section and the third equation determine the direction of the N.A. (directed toward principal axis where M is applied). The same method is used to determine the N.A. when only the component of M toward y is considered.

The principle of superposition can be used to determine stresses in the most general case of unsymmetric bending. Consider first a member with a vertical plane of symmetry, which is subjected to bending couples **M** and **M’** acting in a plane forming an angle θ with the vertical plane.









$$M_z = M\ \cos\theta \qquad M_y = M\ \sin\theta$$

Since the y and z axes are the principal centroidal axes of the cross section, we can use the equation $\sigma_x = -My/I$ to determine the stresses resulting from the application of either of the couples represented by $\mathbf{M}_z$ and $\mathbf{M}_y$:

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

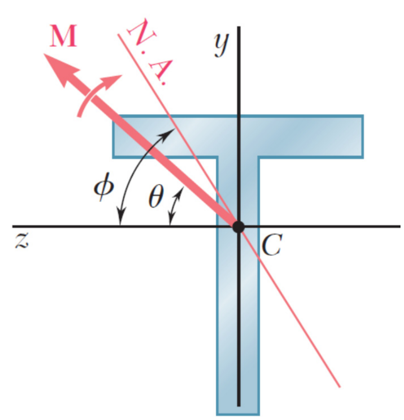
To find the position of neutral axis:

$$0 = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \rightarrow y = \left(\frac{I_z}{I_y}\tan\theta\right) z$$

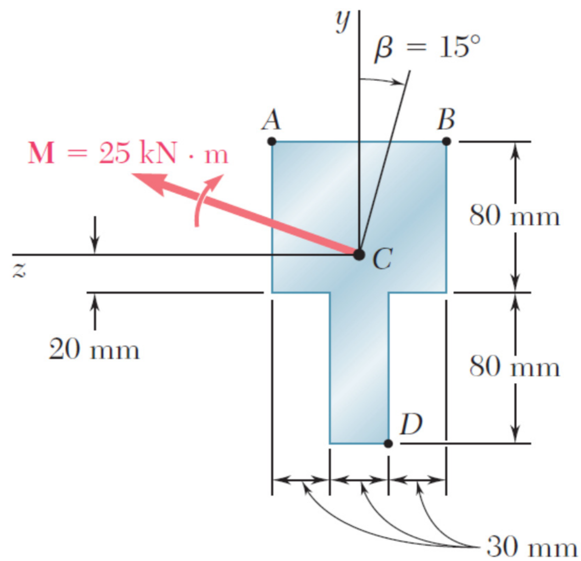
Thus, the angle φ that the neutral axis forms with the z axis is defined by the relation:

$$\tan\varphi = \frac{I_z}{I_y}\tan\theta$$

where θ is the angle that the couple vector ***M*** forms with the same axis. Since I_z and I_y are both positive, φ and θ have the same sign. Furthermore, we note that φ > θ when $I_z > I_y$, and φ < θ when $I_z < I_y$. Thus, the neutral axis is always located between the couple vector ***M*** and the principal axis corresponding to the minimum moment of inertia.



Example 12: The couple **M** is applied to a beam of the cross section shown in a plane forming an angle b with the vertical. Determine the stress at (*a*) point *A*, (*b*) point *B*, (*c*) point *D*.



Decomposing the moment on the principal axes:

$$M_z = 25\ \cos 15° = 24.15\ kNm$$

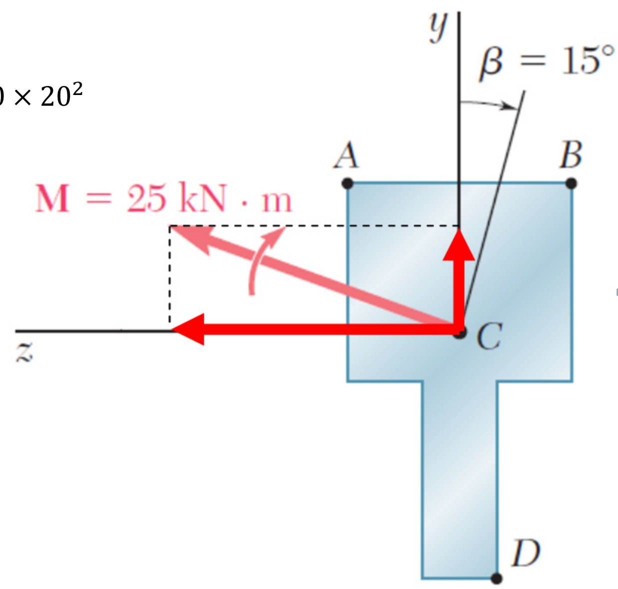
$$M_y = 25\ \sin 15° = 6.47\ kNm$$

Calculating moment of inertia about the principal axes:

$$\bar{y} = \frac{\sum \bar{y}A}{\sum A} = 100\ mm \quad (from\ the\ base)$$

$$I_z = \frac{1}{12}30 \times 80^3 + 30 \times 80 \times 60^2 + \frac{1}{12}90 \times 80^3 + 90 \times 80 \times 20^2$$
$$= 16\ 640\ 000\ mm^4$$

$$I_y = \frac{1}{12}80 \times 30^3 + \frac{1}{12}80 \times 90^3 = 5\ 040\ 000\ mm^4$$



Calculating stress at point A:

$$\sigma_A = \frac{M_z c_y}{I_z} + \frac{M_y c_z}{I_y}$$
$$= -\frac{24.15 \times 10^6\ Nmm \times 60\ mm}{16\ 640\ 000\ mm^4} + \frac{6.47 \times 10^6\ Nmm \times 45\ mm}{5\ 040\ 000\ mm^4} = -29.3\ MPa$$

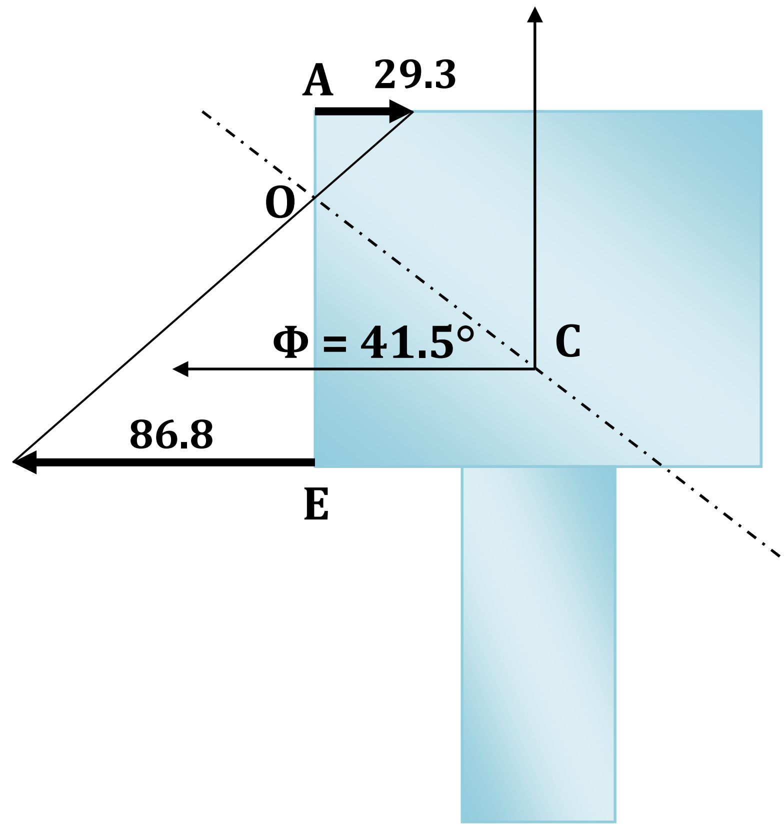
Calculating stress at point B:

$$\sigma_B = \frac{M_z c_y}{I_z} + \frac{M_y c_z}{I_y}$$
$$= -\frac{24.15 \times 10^6\ Nmm \times 60\ mm}{16\ 640\ 000\ mm^4} - \frac{6.47 \times 10^6\ Nmm \times 45\ mm}{5\ 040\ 000\ mm^4} = -144.8\ MPa$$

Calculating stress at point D:

$$\sigma_D = \frac{M_z c_y}{I_z} + \frac{M_y c_z}{I_y}$$
$$= +\frac{24.15 \times 10^6\ Nmm \times 100\ mm}{16\ 640\ 000\ mm^4} - \frac{6.47 \times 10^6\ Nmm \times 15\ mm}{5\ 040\ 000\ mm^4} = +125.7\ MPa$$

Example 12 (continued):



Finding position of the neutral axis (Method 1):

We know that the neutral axis passes through point C where stress is zero. If we can find a second point where stress is zero we can find the position of neutral axis.

$$\sigma_E = \frac{M_z c_y}{I_z} + \frac{M_y c_z}{I_y}$$
$$= +\frac{24.15 \times 10^6\ Nmm \times 20\ mm}{16\ 640\ 000\ mm^4}$$
$$+\frac{6.47 \times 10^6\ Nmm \times 45\ mm}{5\ 040\ 000\ mm^4}$$
$$= +86.8\ MPa$$

As the stress at point A is negative and at point E positive there should be a point in between where the stress is zero. As the variation of stress is linear this point of zero stress can be found easily:

$$\frac{29.3}{86.8} = \frac{AO}{EO} \rightarrow \frac{29.3 + 86.8}{86.8} = \frac{Ao + EO}{EO}$$

$$\frac{116.1}{86.8} = \frac{80}{EO} \rightarrow EO = 59.8\ mm$$

$$\varphi = arctg\left(\frac{59.8 - 20}{45}\right) = 41.5°$$

Finding position of the neutral axis (Method 2):

$$\tan\varphi = \frac{I_z}{I_y}\tan\theta = \frac{16\ 640\ 000\ mm^4}{5\ 040\ 000\ mm^4}\tan 15° \rightarrow \varphi = 41.5°$$

Method 3:

Best Method to determine NA:

$$\sigma_A = \frac{M_z c_y}{I_z} + \frac{M_y c_z}{I_y}$$
$$= -\frac{24.15 \times 10^6\ Nmm \times Y}{16\ 640\ 000\ mm^4}$$
$$+\frac{6.47 \times 10^6\ Nmm \times Z}{5\ 040\ 000\ mm^4} = 0$$

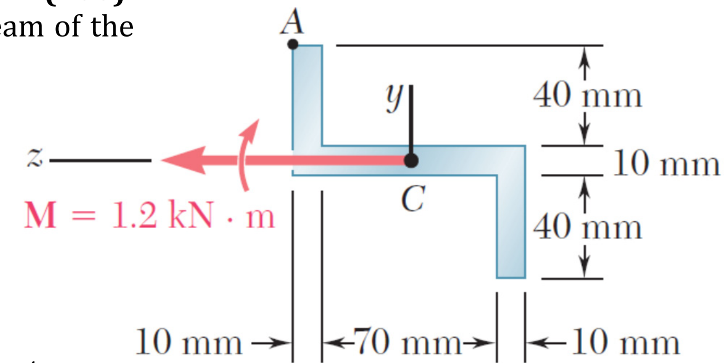
Y = 0.8845 Z (NA equation)

OR

$$\sigma_B = \frac{M_z c_y}{I_z} + \frac{M_y c_z}{I_y}$$
$$= -\frac{24.15 \times 10^6\ Nmm \times Y}{16\ 640\ 000\ mm^4}$$
$$-\frac{6.47 \times 10^6\ Nmm \times (-Z)}{5\ 040\ 000\ mm^4} = 0$$

Y = 0.8845 Z (NA equation)

Example 13: The couple **M** is applied to a beam of the cross section shown. Find stress at point A.



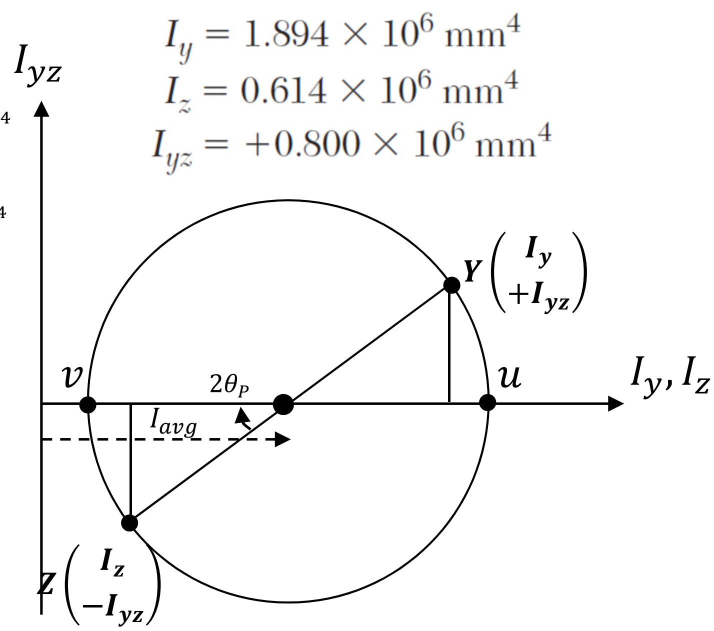
Calculating section properties:

$$I_y = \frac{1}{12}10 \times 90^3 + 2\left(\frac{1}{12}40 \times 10^3 + 40 \times 10 \times 40^2\right)$$
$$= 1\,894\,167\ mm^4$$

$$I_z = \frac{1}{12}90 \times 10^3 + 2\left(\frac{1}{12}10 \times 40^3 + 40 \times 10 \times 25^2\right)$$
$$= 614\,166.7\ mm^4$$

$$I_{yz} = \int yz\, dA = (0)(0)(10 \times 90) + (25)(40)(40 \times 10)$$
$$+ (-25)(-40)(40 \times 10) = 800\,000\ mm^4$$

Finding principal axes of the section (Mohr's circle):



$$I_v = I_{avg} - R = \frac{I_y + I_z}{2} - \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}{}^2} = 229\,666.8\ mm^4$$

$$I_u = I_{avg} + R = \frac{I_y + I_z}{2} + \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}{}^2} = 227\,8667\ mm^4$$

$$\tan 2\theta_P = \frac{I_{yz}}{\frac{I_y - I_z}{2}} = \frac{800\,000}{\frac{1\,894\,167 - 614\,166.7}{2}} = 0.625 \to 2\theta_P$$
$$= 51.34° \to \theta_P = -25.67°$$

Decomposing the moment on the principal axes:

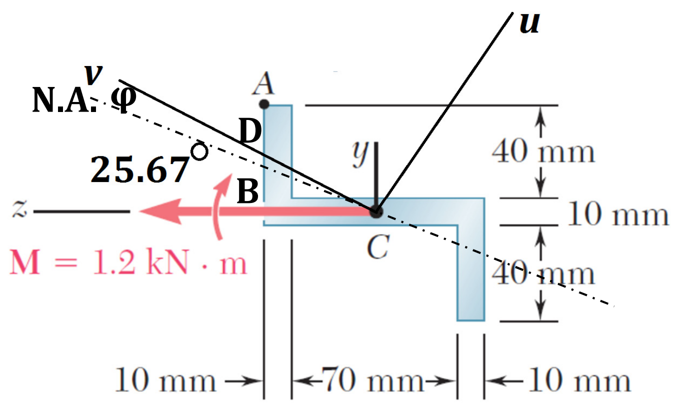
$$M_v = 1.2\ \cos(-25.67°) = 1.0816\ kNm$$

$$M_u = 1.2\ \sin(-25.67°) = -0.5198\ kNm$$

Finding coordinate of point A in uv system:

$$\sigma_A = \frac{M_v u_A}{I_v} + \frac{M_u v_A}{I_u}$$

$$\begin{pmatrix} v_A \\ u_A \end{pmatrix} = \begin{bmatrix} \cos\theta_P & -\sin\theta_P \\ \sin\theta_P & \cos\theta_P \end{bmatrix} \begin{pmatrix} z_A \\ y_A \end{pmatrix}$$
$$= \begin{bmatrix} \cos(-25.67°) & -\sin(-25.67°) \\ \sin(-25.67°) & \cos(-25.67°) \end{bmatrix} \begin{pmatrix} 45 \\ 45 \end{pmatrix}$$
$$= \begin{pmatrix} 60.05 \\ 21.07 \end{pmatrix} mm$$



Calculating stress at point A:

$$\sigma_A = \frac{M_v u_A}{I_v} + \frac{M_u v_A}{I_u}$$
$$= -\frac{(1.0816 \times 10^6\ Nmm)(21.07\ mm)}{229\,666.8\ mm^4} - \frac{(0.5198 \times 10^6\ Nmm)(60.05\ mm)}{227\,8667\ mm^4}$$
$$= -112.97\ MPa$$

Calculating stress at point B:

$$\begin{pmatrix} v_B \\ u_B \end{pmatrix} = \begin{bmatrix} \cos\theta_P & -\sin\theta_P \\ \sin\theta_P & \cos\theta_P \end{bmatrix} \begin{pmatrix} z_B \\ y_B \end{pmatrix}$$
$$= \begin{bmatrix} \cos(-25.67°) & -\sin(-25.67°) \\ \sin(-25.67°) & \cos(-25.67°) \end{bmatrix} \begin{pmatrix} 45 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 40.56 \\ -19.5 \end{pmatrix} mm$$

$$\sigma_B = \frac{M_v u_B}{I_v} + \frac{M_u v_B}{I_u}$$
$$= +\frac{(1.0816 \times 10^6\ Nmm)(19.5\ mm)}{229\,666.8\ mm^4} - \frac{(0.5198 \times 10^6\ Nmm)(40.56\ mm)}{227\,8667\ mm^4}$$
$$= +80.3\ MPa$$

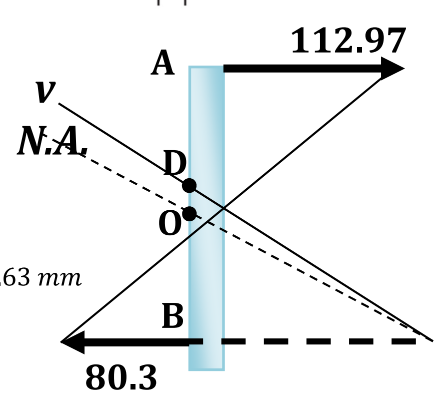
$$\frac{112.97}{80.3} = \frac{AO}{BO} \to$$

$$\frac{112.97 + 80.3}{80.3} = \frac{45}{BO}$$

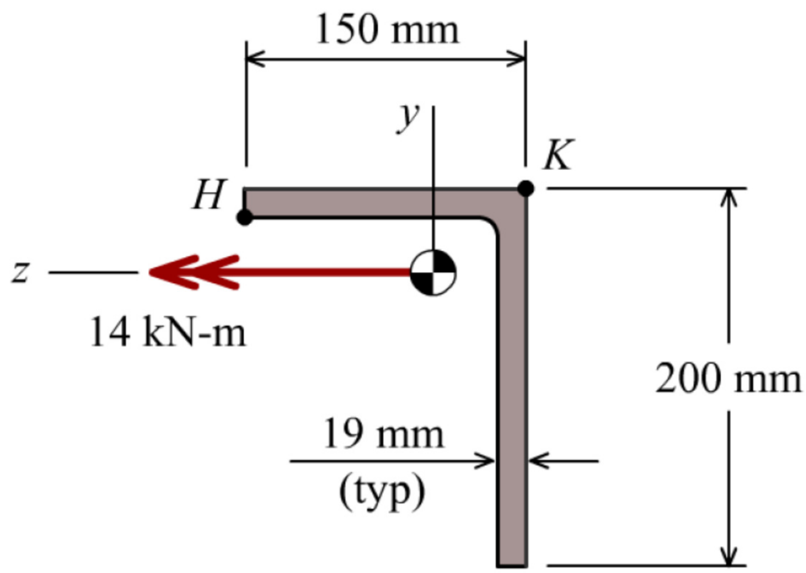
$$BO = 18.7\ mm$$

$$BD = 45\ tan25.67° = 21.63\ mm$$

$$\tan\varphi = \frac{I_v}{I_u}\tan(-25.67°)$$
$$= \frac{229\,666.8}{227\,8667}\tan(-25.67°)$$
$$\to \varphi = -2.77°$$



TBR 10: The moment acting on the cross section of the unequal-leg angle has a magnitude of 14 kNm and is oriented as shown. Determine: (a) the bending stress at point *H*, (b) the bending stress at point *K*, (c) the maximum tension and the maximum compression bending stresses in the cross section, (d) the orientation of the neutral axis relative to the +*z* axis. Show its location on a sketch of the cross section.



Answer: Centroid location: 64.18 mm (from bottom of shape to centroid) and 39.18 mm from right edge of shape to centroid. Moment of inertia about the *z* axis (Iz): 25,059,086.23 mm^4. Moment of inertia about the *y* axis (Iy): 12,133,386.23 mm^4. Product of inertia about the centroidal axes (Iyz): 10,207,907.81 mm^4. Bending stress at K: -82.6 MPa compression. Maximum tension and compression bending stresses: 101 MPa and -82.6 MPa. Orientation of neutral axis is shown.