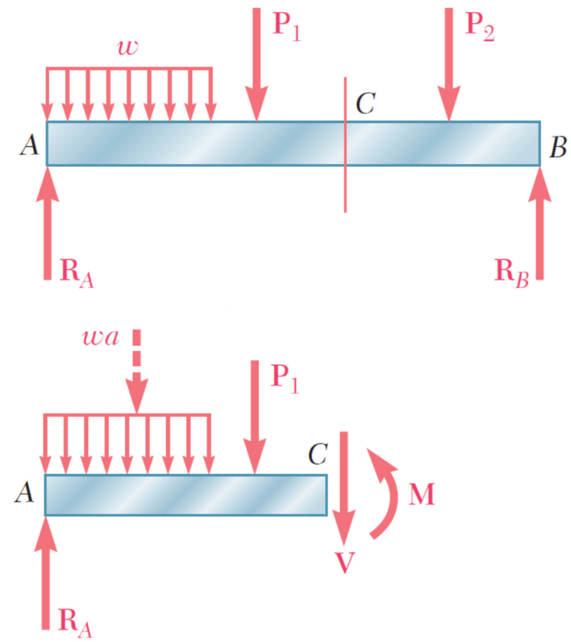
CHAPTER 5: EQUILIBRIUM OF BEAMS



Our objective in this chapter is to determine the diagram of internal moments (M) and shear forces (V) in beams supporting transverse loading. The bending couple M creates *normal stresses* in the cross section, while the shear force V creates *shearing stresses* in that section. In most cases the dominant criterion in the design of a beam for strength is the maximum value of the normal stress in the beam. The determination of the normal stresses ($\sigma_x = Mc/I$) in a beam will be the subject of this chapter, while shearing stresses will be discussed in the next chapter.

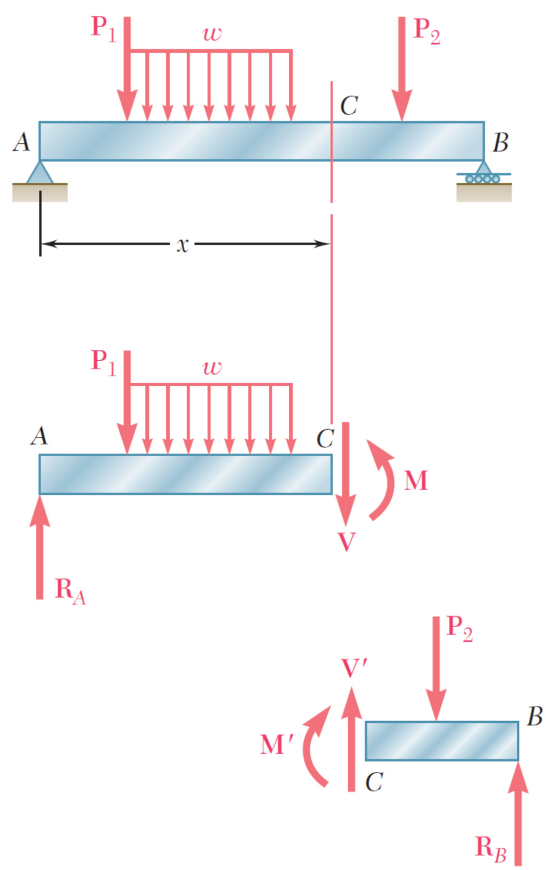
Shear and Bending-Moment Diagrams

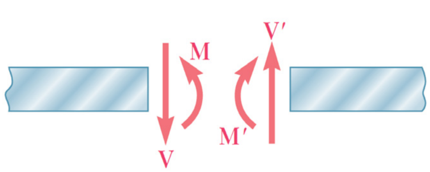
There are three methods to determine the diagram of internal shear forces and moments in a beam under transverse loading:

1. Equilibrium Method
2. Load-Shear-Moment Relationship Method
3. Singularity Function Method

Equilibrium Method

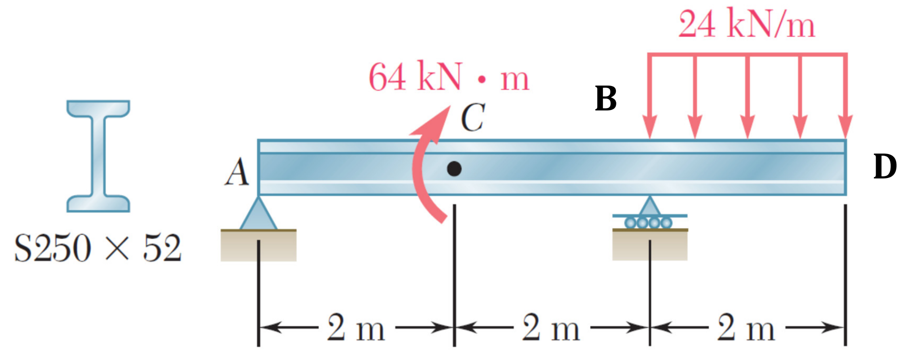
The shear and bending-moment diagrams will be obtained by determining the values of *V* and *M* at selected points of the beam. These values will be found in the usual way, i.e., by passing a section through the point where they are to be determined and considering the **equilibrium** of the portion of beam located on either side of the section. The shear V and the bending moment M at a given point of a beam are said to be positive when the internal forces and couples acting on each portion of the beam are directed as shown.





Internal Forces
(positive shear and positive bending moment)

Example 1: Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.



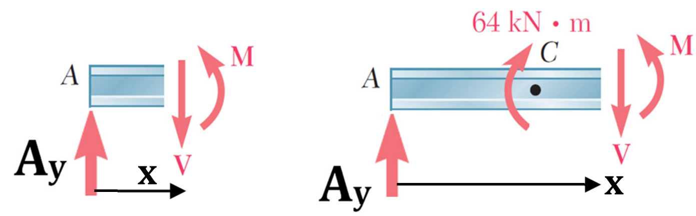
Finding support reaction forces:

$\overset{+}{\rightarrow} \sum F_x = 0 \rightarrow A_x = 0.$

$+\uparrow \sum F_y = 0 \rightarrow A_y + B_y - 24 \times 2\ kN = 0.$

$\curvearrowright \sum M_A = 0 \rightarrow 64 - B_y \times 4 + (24 \times 2) \times 5 = 0.$

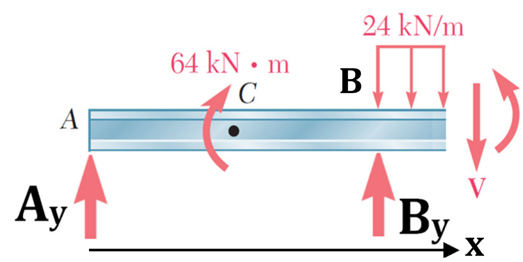
$B_y = 76\ kN, \qquad A_y = -28\ kN$



Equilibrium of part AC:

$+\uparrow \sum F_y = 0 \rightarrow V = A_y = -28\ kN$

$\curvearrowright \sum M = 0 \rightarrow -28x - M = 0. \rightarrow M = -28x$



Equilibrium of part CB:

$+\uparrow \sum F_y = 0 \rightarrow V = A_y = -28\ kN$

$\curvearrowright \sum M = 0 \rightarrow -28x + 64 - M = 0.$

$\rightarrow M = -28x + 64$

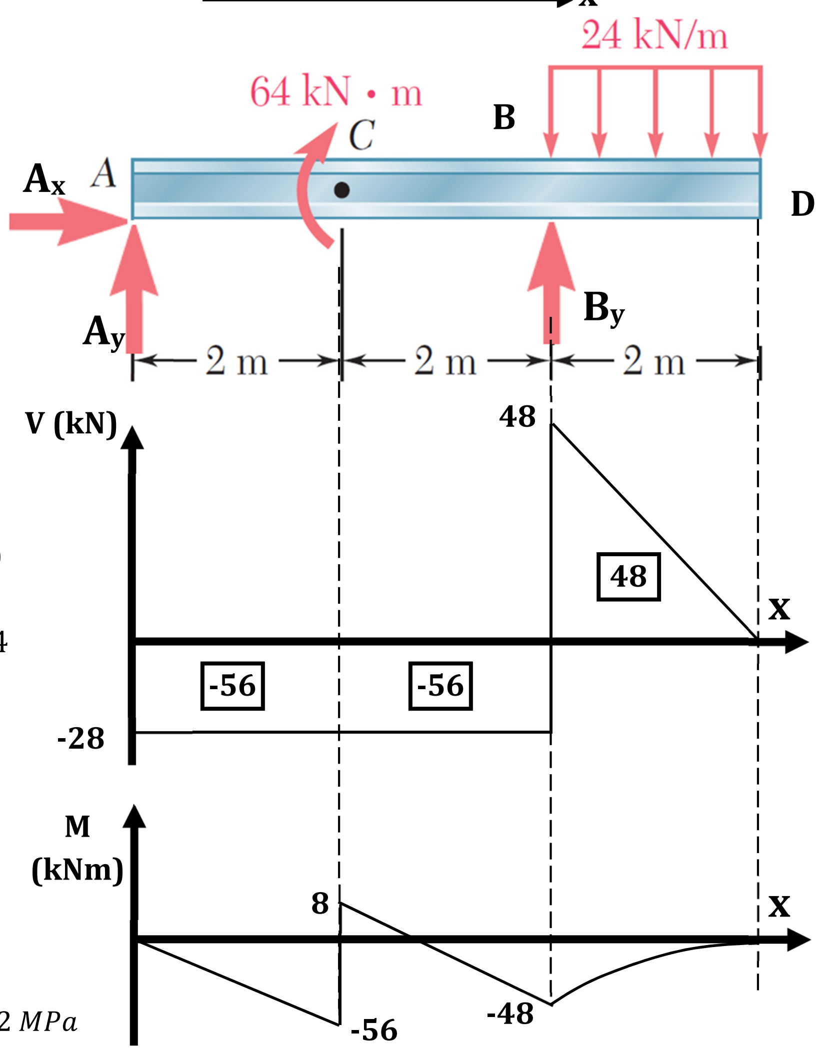
Equilibrium of part BD:

$+\uparrow \sum F_y = 0 \rightarrow A_y + B_y - 24(x-4) - V = 0$

$V = -28 + 76 - 24x + 96 \rightarrow V = -24x + 144$

$$\curvearrowright \sum M = 0 \rightarrow -28x + 64 + 76(x-4) - \frac{24(x-4)(x-4)}{2} - M = 0.$$

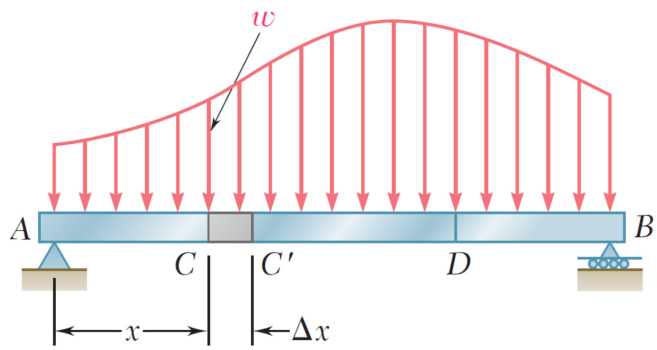
$\rightarrow M = -12x^2 + 144x - 432$



Maximum normal stress:

$$\sigma = \frac{Mc}{I} = \pm \frac{(56\,000\,000\ Nmm)(254/2)}{61.2 \times 10^6\ mm^4} = \pm 116.2\ MPa$$

Load-Shear-Moment Relationship Method



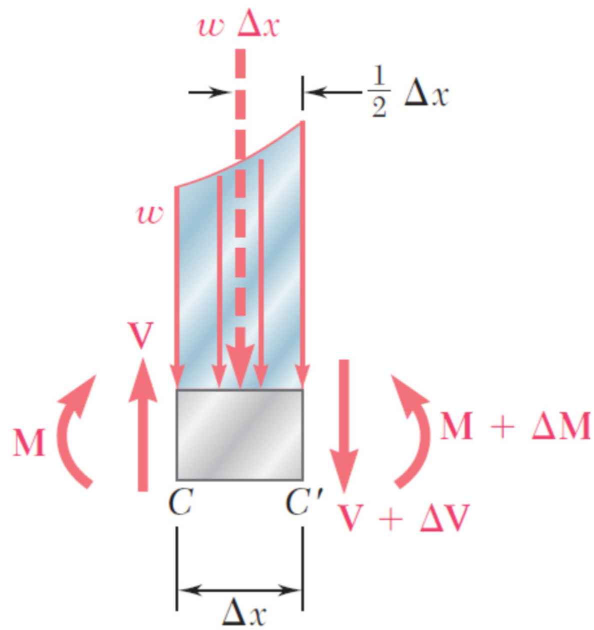
The construction of the shear diagram and, especially, of the bending-moment diagram will be greatly facilitated if certain relations existing among load, shear, and bending moment are taken into consideration.

$+\uparrow \sum F_y = 0 \rightarrow V - (V + \Delta V) - wx = 0 \rightarrow$

$$\Delta V = -w\,\Delta x \rightarrow \frac{dV}{dx} = -w \rightarrow$$

$$V_D - V_C = -\int_{x_C}^{x_D} w\,dx \rightarrow$$

$V_D - V_C$ = - (area under load curve between C and D)



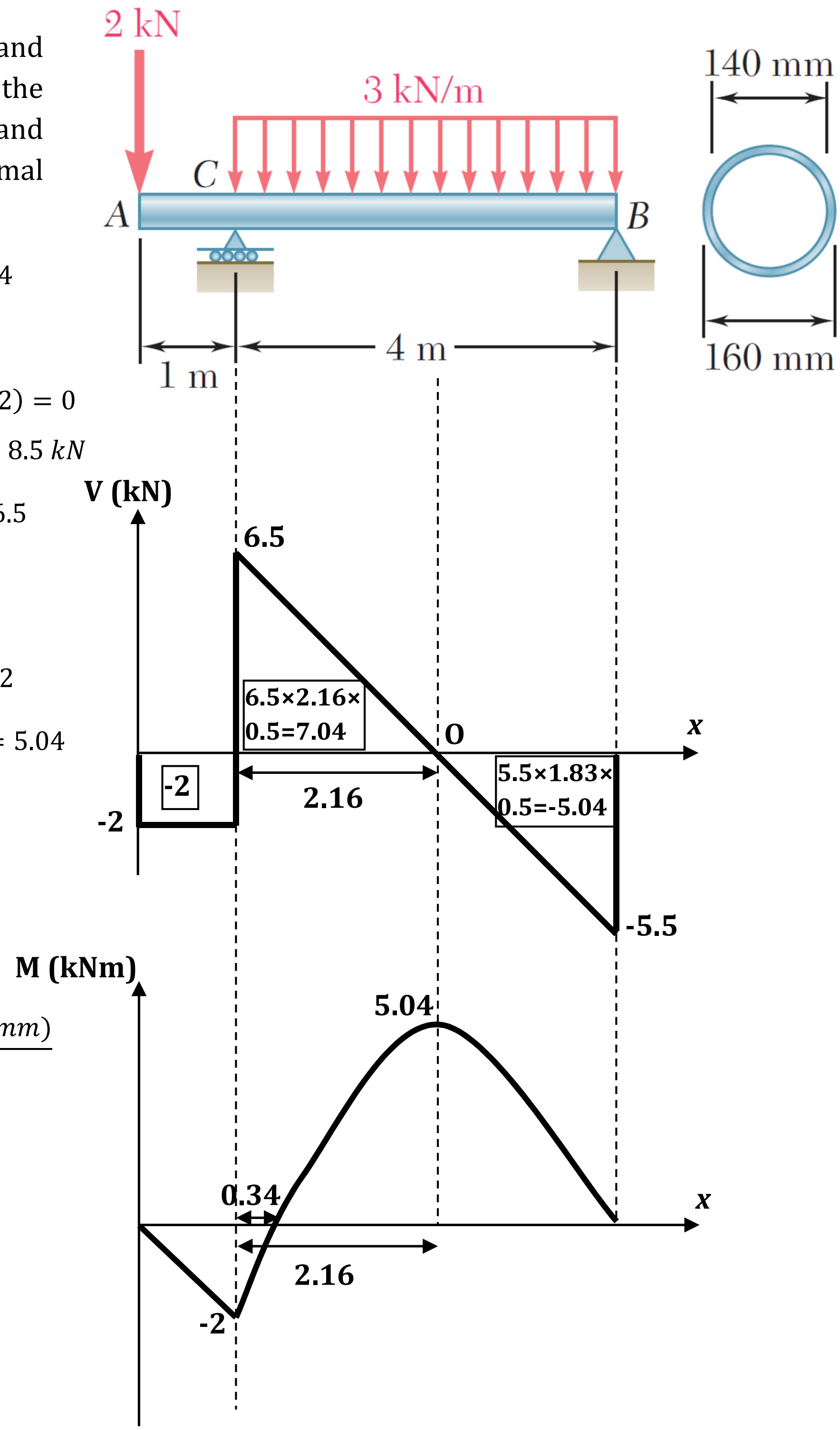
$$+\curvearrowleft \sum M_{\acute{C}} = 0 \rightarrow (M + \Delta M) - M + w\Delta x\frac{\Delta x}{2} - V\Delta x = 0$$

$$\Delta M = V\Delta x - \frac{w\Delta x^2}{2} \approx V\Delta x \rightarrow \frac{dM}{dx} = V$$

$$M_D - M_C = \int_{x_C}^{x_D} V\,dx \rightarrow$$

$M_D - M_C$ = (area under shear curve between C and D)

Example 2: Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.



$$+\uparrow\sum F_y = 0 \rightarrow R_C + R_B = 2 + 3\times 4$$
$$= 14\ kNm$$

$$\curvearrowleft\sum M_C = 0 \rightarrow 2(1) + R_B(4) - 12(2) = 0$$
$$\rightarrow R_B = 5.5\ kN, R_C = 8.5\ kN$$

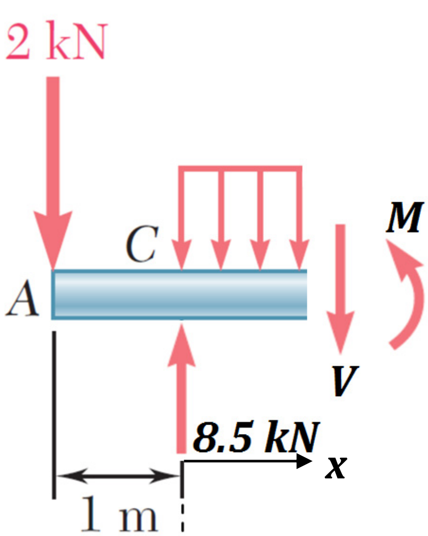
$$V_B - V_C = -(3\times 4) \rightarrow V_B = -12 + 6.5$$
$$= -5.5\ kN$$

$$M_C - M_A = -2 \rightarrow M_C = -2 \pm 0 = -2$$

$$M_O - M_C = 7.04 \rightarrow M_O = 7.04 - 2 = 5.04$$

$$At\ point\ O: \frac{dM}{dx} = V = 0$$

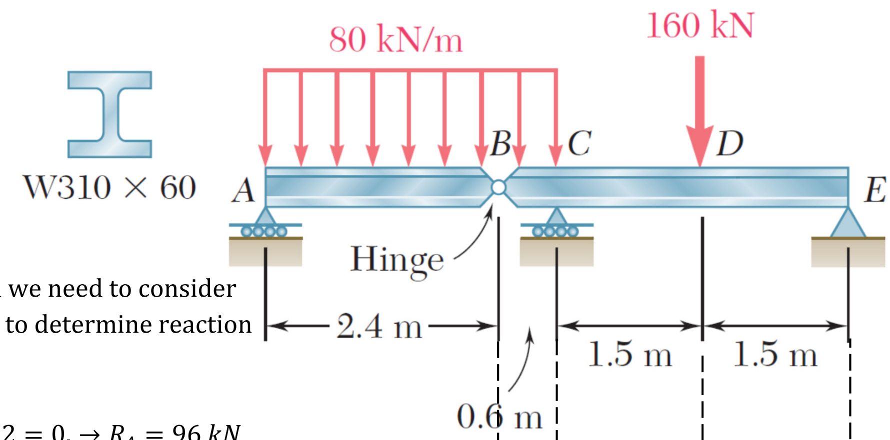
$$\sigma = \frac{Mc}{I} = \pm\frac{(5.04\times 10^6\ Nmm)(80\ mm)}{\frac{\pi}{4}(80^4 - 70^4)\ mm^4}$$
$$= \pm 30.28\ MPa$$



$$\boldsymbol{M + 3x\frac{x}{2} + 2(1+x) - 8.5x = 0}$$

$$\boldsymbol{M = 0 \rightarrow x = 0.34\ m\ and\ x = 4m}$$

Example 3: Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.



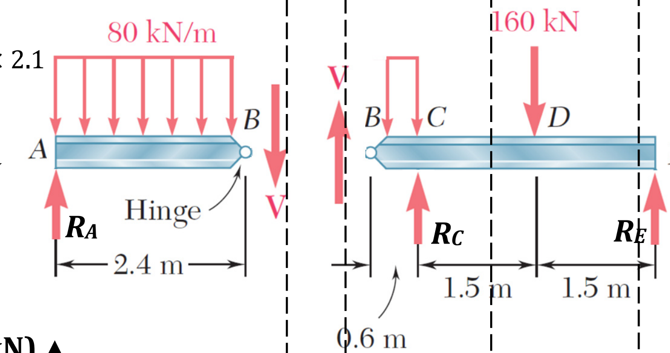
The system is statically indeterminate and we need to consider equilibrium of parts AB and BE separately to determine reaction forces:

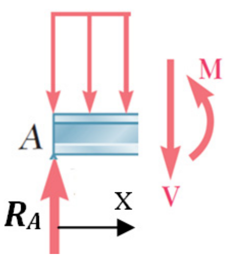
$$AB: \curvearrowright \sum M_B = 0 \rightarrow 2.4R_A - 80 \times 2.4 \times 1.2 = 0. \rightarrow R_A = 96\ kN$$

$$BE: \curvearrowleft \sum M_B = 0 \rightarrow 3.6R_E + 0.6R_C - 80 \times 0.6 \times 0.3 - 160 \times 2.1 = 0. \rightarrow 3.6R_E + 0.6R_C = 350.4\ kN$$

$$AE: +\uparrow \sum F_y = 0 \rightarrow R_A + R_C + R_E = 160 + 80 \times 3 = 400\ kN$$

$$\rightarrow R_C + R_E = 304\ kN \rightarrow R_E = 56\ kN, \qquad R_C = 248\ kN$$

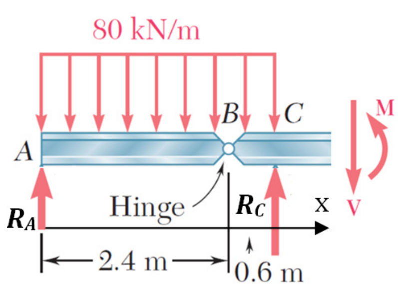




Part AC:

$$V = 96 - 80x$$

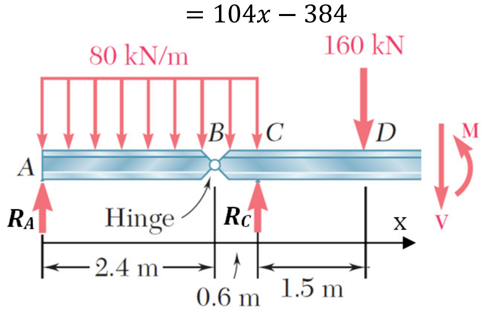
$$M = 96x - 40x^2$$



Part CD:

$$V = 96 + 248 - 80 \times 3 = 104$$

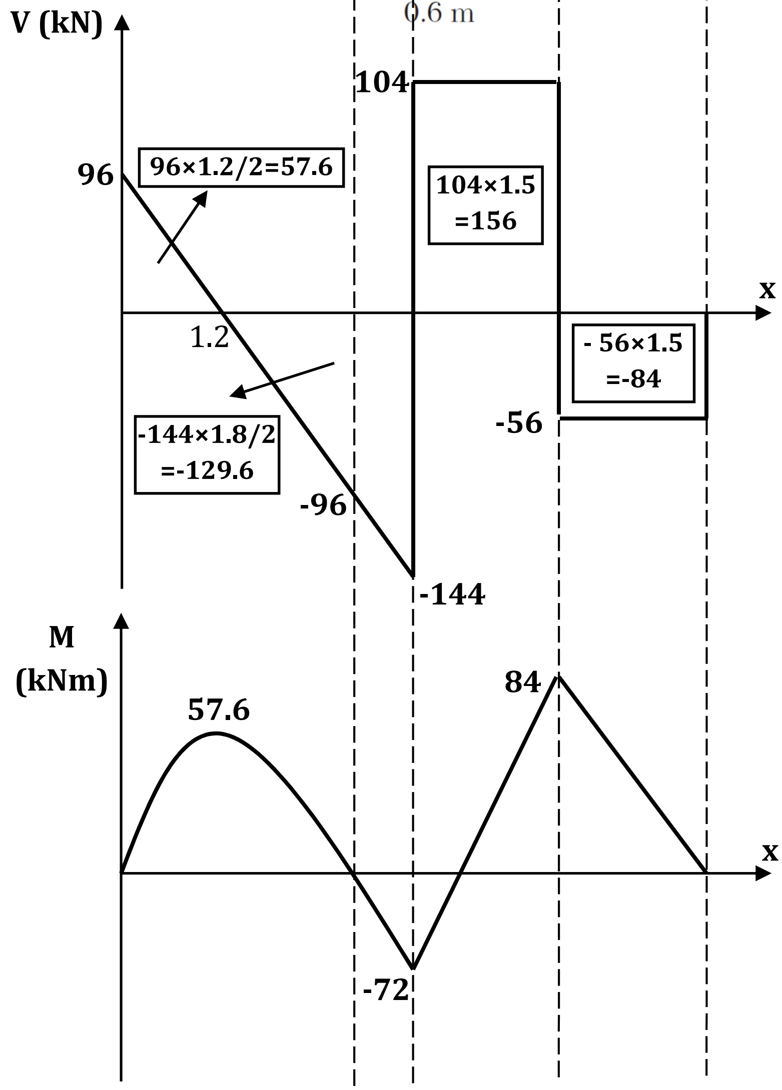
$$M = 96x + 248(x-3) - 80 \times 3 \times (x - 1.5) = 104x - 384$$



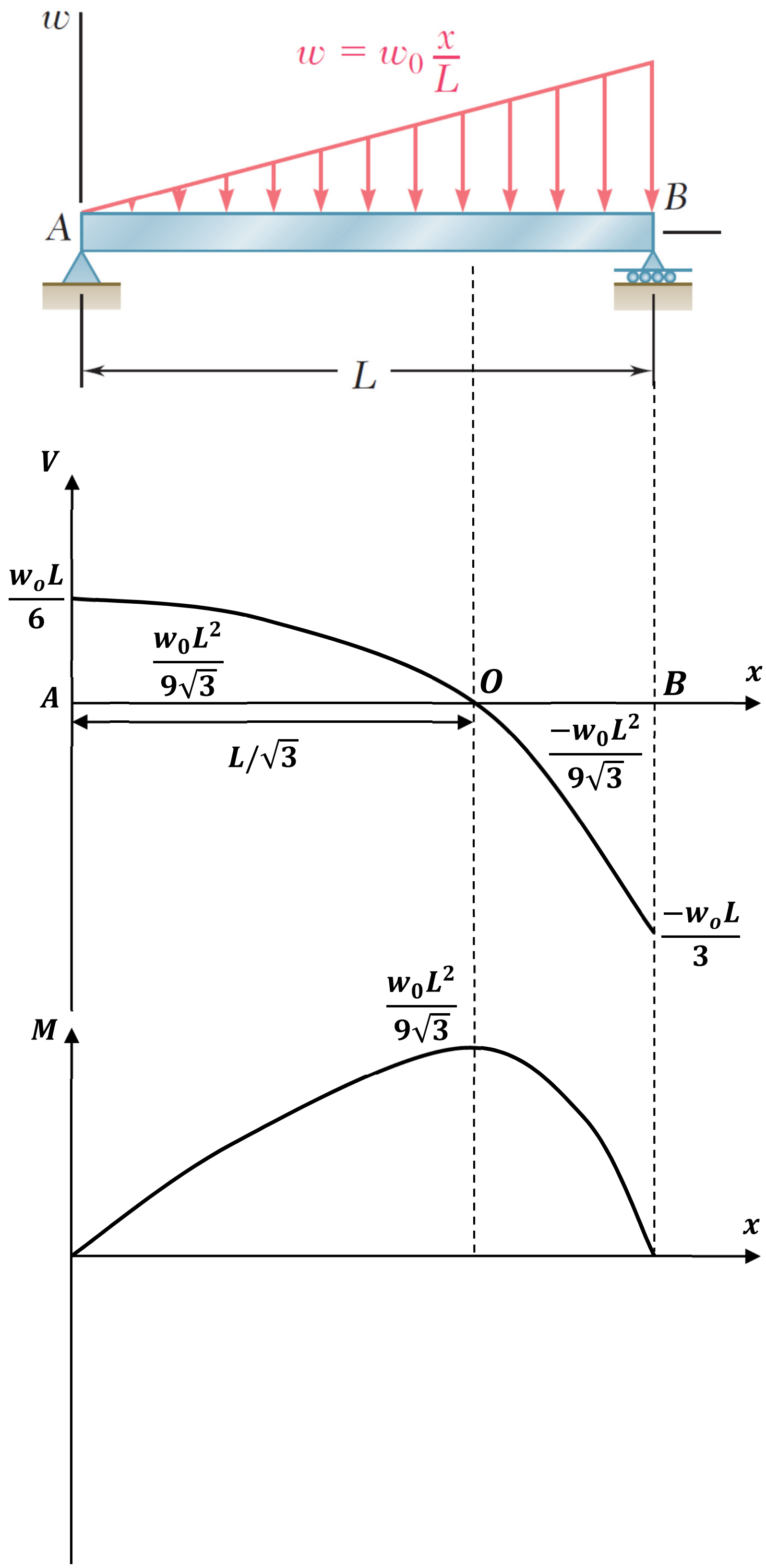
Part DE:

$$V = 96 + 248 - 80 \times 3 - 160 = -56$$

$$M = 96x + 248(x-3) - 80 \times 3 \times (x - 1.5) - 160(x - 4.5) = -56x + 336$$



TBR 1: Draw the shear and bending-moment diagrams for the beam and loading shown using **Load-Shear-Moment Relationship Method.**



From Statics:

$$+\uparrow \sum F_y = 0 \rightarrow R_A + R_B = \frac{w_o L}{2}$$

$$\curvearrowleft \sum M_A = 0 \rightarrow R_B L - \frac{w_o L}{2}\frac{2L}{3} = 0$$

$$R_A = \frac{w_o L}{6}, R_B = \frac{w_o L}{3}$$

Load-Shear Relationship

$$V_B - V_A = -\frac{w_o L}{2} \rightarrow V_B = \frac{w_o L}{6} - \frac{w_o L}{2}$$
$$= -\frac{w_o L}{3}$$

At point A: $\frac{dV}{dx} = -\omega = 0$

$$V = \frac{w_0}{6L}(-3x^2 + L^2) \rightarrow V = 0\ at\ x = L/\sqrt{3}$$

Area under AO:

$$\int_0^{L/\sqrt{3}} \frac{w_0}{6L}(-3x^2 + L^2)dx = \frac{w_0 L^2}{9\sqrt{3}}$$

Area under OB: $-\frac{w_0 L^2}{9\sqrt{3}}$

Shear-moment Relationship

$$M_O - M_A = \frac{w_0 L^2}{9\sqrt{3}} \rightarrow M_O = \frac{w_0 L^2}{9\sqrt{3}}$$

$$M = \frac{w_0}{6}\left(\frac{x^3}{L} + Lx\right)$$

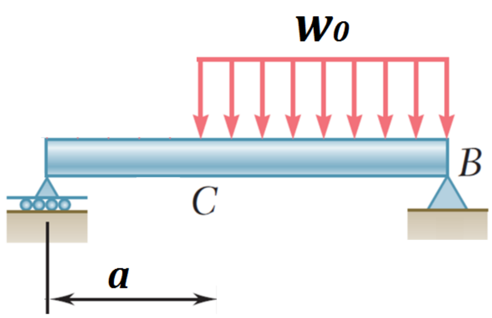
Singularity Function Method

$$\langle x-a\rangle^n = \begin{cases}(x-a)^n & when \;\; x \geq a \\ 0 & when \;\; x < a\end{cases} \qquad n \geq 0$$

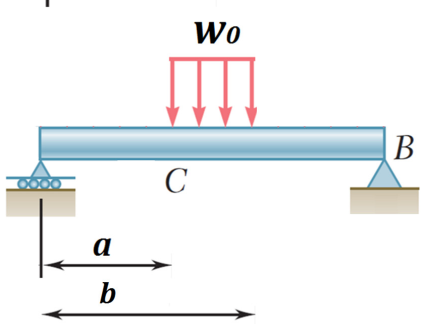
$$\langle x-a\rangle^0 = \begin{cases}(x-a)^0 & when \;\; x \geq a \\ 0 & when \;\; x < a\end{cases} = \begin{cases}1 & when \;\; x \geq a \\ 0 & when \;\; x < a\end{cases}$$

$$\int \langle x-a\rangle^n dx = \frac{1}{n+1}\langle x-a\rangle^{n+1} \qquad for \quad n \geq 0$$

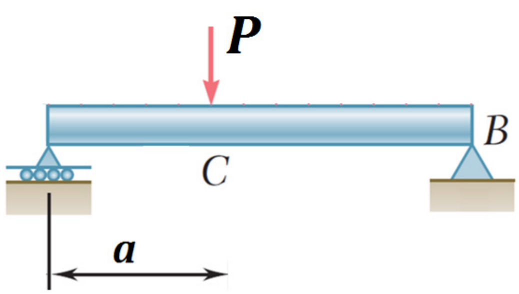
$$\frac{d}{dx}\langle x-a\rangle^n = n\langle x-a\rangle^{n-1} \qquad for \quad n \geq 1$$



$$w(x) = w_0\langle x-a\rangle^0 = \begin{cases}w_0 & when \;\; x \geq a \\ 0 & when \;\; x < a\end{cases}$$



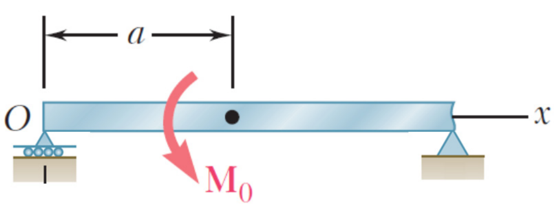
$$w(x) = w_0\langle x-a\rangle^0 - w_0\langle x-b\rangle^0$$



$$\langle x-a\rangle^{-1} = \begin{cases}0 & when \;\; x \neq a \\ 1 & when \;\; x = a\end{cases}$$

$$w(x) = P\langle x-a\rangle^{-1} = \begin{cases}0 & when \;\; x \neq a \\ P & when \;\; x = a\end{cases}$$

(-1 is considered so that unit for *w* becomes N/m)



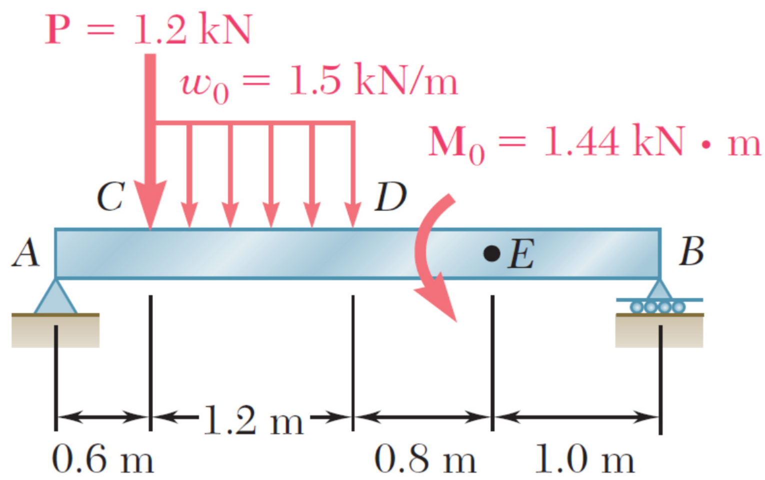
$$\langle x-a\rangle^{-2} = \begin{cases}0 & when \;\; x \neq a \\ 1 & when \;\; x = a\end{cases}$$

$$w(x) = M\langle x-a\rangle^{-2} = \begin{cases}0 & when \;\; x \neq a \\ M & when \;\; x = a\end{cases}$$

(-2 is considered so that unit for *w* becomes N/m)

$$\int \langle x-a\rangle^n dx = \langle x-a\rangle^{n+1} \qquad for \quad n < 0$$

Example 4: For the beam and loading shown and using singularity functions, express the shear and bending moment as functions of the distance *x* from the support at *A*.



$$\circlearrowleft \sum M_B = 0 \rightarrow 1.44 + 1.5 \times 1.2 \times 2.4 + 1.2 \times 3 - R_A(3.6) = 0 \rightarrow R_A = 2.6\ kN$$

$$w(x) = -2.6\langle x-0\rangle^{-1} + 1.2\langle x-0.6\rangle^{-1} + 1.5\langle x-0.6\rangle^{0} - 1.5\langle x-1.8\rangle^{0} + M_0\langle x-2.6\rangle^{-2}$$

$$V(x) = -\int w dx = 2.6\langle x-0\rangle^{0} - 1.2\langle x-0.6\rangle^{0} - \frac{1.5}{1}\langle x-0.6\rangle^{1} + \frac{1.5}{1}\langle x-1.8\rangle^{1} - M_0\langle x-2.6\rangle^{-1}$$

$$M(x) = \int V(x)dx = \frac{2.6}{1}\langle x-0\rangle^{1} - \frac{1.2}{1}\langle x-0.6\rangle^{1} - \frac{1.5}{2}\langle x-0.6\rangle^{2} + \frac{1.5}{2}\langle x-1.8\rangle^{2} - M_0\langle x-2.6\rangle^{0}$$

$$V(x \approx 0) = 2.6\langle \varepsilon-0\rangle^{0} - 1.2\langle \varepsilon-0.6\rangle^{0} - \frac{1.5}{1}\langle \varepsilon-0.6\rangle^{1} + \frac{1.5}{1}\langle \varepsilon-1.8\rangle^{1} - M_0\langle \varepsilon-2.6\rangle^{-1} = 2.6 - 0 - 0 + 0 - 0 = 2.6\ kN = R_A$$

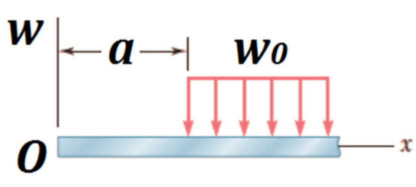
$$M(x \approx 0) = \frac{2.6}{1}\langle \varepsilon-0\rangle^{1} - \frac{1.2}{1}\langle \varepsilon-0.6\rangle^{1} - \frac{1.5}{2}\langle \varepsilon-0.6\rangle^{2} + \frac{1.5}{2}\langle \varepsilon-1.8\rangle^{2} - M_0\langle \varepsilon-2.6\rangle^{0} = 2.6\varepsilon = 0$$

V_E and M_E should be calculated in a similar way.

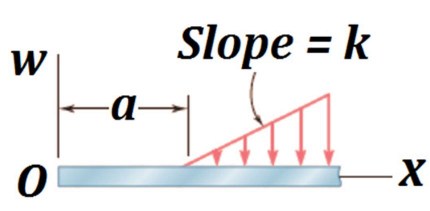
Finding shear and moment at point D for example:

$$V_D = V(x=1.8) = 2.6\langle 1.8-0\rangle^{0} - 1.2\langle 1.8-0.6\rangle^{0} - \frac{1.5}{1}\langle 1.8-0.6\rangle^{1} + \frac{1.5}{1}\langle 1.8-1.8\rangle^{1} - M_0\langle 1.8-2.6\rangle^{-1}$$
$$= 2.6 - 1.2 - 1.5(1.8-0.6) + 0 - 0 = -0.4\ kN$$

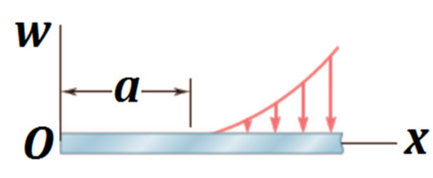
$$M_D = M(x=1.8) = \frac{2.6}{1}\langle 1.8-0\rangle^{1} - \frac{1.2}{1}\langle 1.8-0.6\rangle^{1} - \frac{1.5}{2}\langle 1.8-0.6\rangle^{2} + \frac{1.5}{2}\langle 1.8-1.8\rangle^{2} - M_0\langle 1.8-2.6\rangle^{0}$$
$$= 2.6(1.8) - 1.2(1.8-0.6) - \frac{1.5}{2}(1.8-0.6)^2 + 0 - 0 = 2.16\ kN$$



$$w(x) = w_0\langle x-a\rangle^0$$

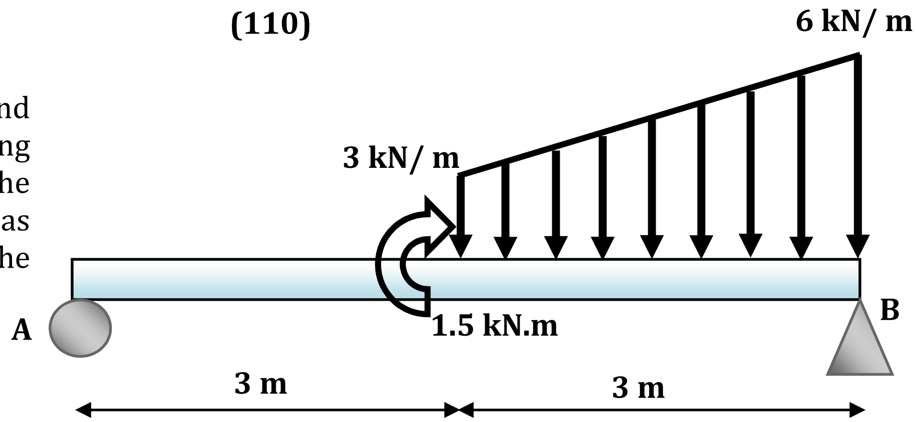


$$w(x) = k\langle x-a\rangle^1$$



$$w(x) = k\langle x-a\rangle^n$$

Example 5: For the beam and loading shown and using singularity functions, express the shear and bending moment as functions of the distance *x* from the support at *A*.



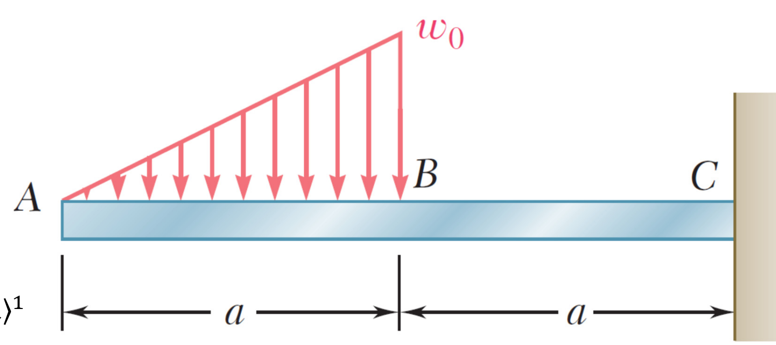
From Statics: $R_A = 2.75$ kN

$$w(x) = -2.75\langle x-0\rangle^{-1} - 1.5\langle x-3\rangle^{-2} + 3\langle x-3\rangle^{0} + 1\langle x-3\rangle^{1}$$

$$V(x) = 2.75\langle x\rangle^{0} + 1.5\langle x-3\rangle^{-1} - \frac{3}{1}\langle x-3\rangle^{1} - \frac{1}{2}\langle x-3\rangle^{2}$$

$$M(x) = \frac{2.75}{1}\langle x\rangle^{1} + 1.5\langle x-3\rangle^{0} - \frac{3}{2}\langle x-3\rangle^{2} - \frac{1}{6}\langle x-3\rangle^{3}$$

Example 6: For the beam and loading shown and using singularity functions, express the shear and bending moment as functions of the distance *x* from the support at *A*.

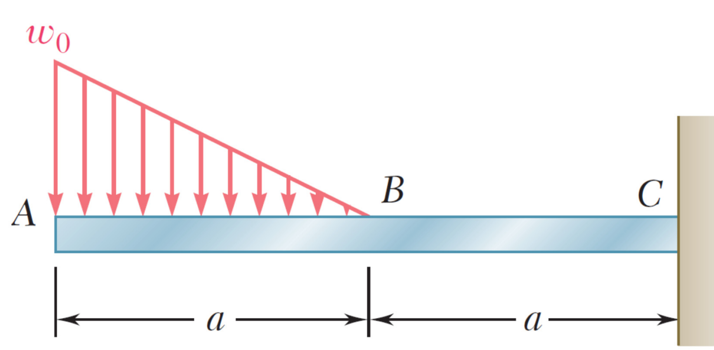


$$w(x) = \frac{w_0}{a}\langle x-0\rangle^{1} - w_0\langle x-a\rangle^{0} - \frac{w_0}{a}\langle x-a\rangle^{1}$$

$$V(x) = -\frac{w_0}{2a}\langle x\rangle^{2} + \frac{w_0}{1}\langle x-a\rangle^{1} + \frac{w_0}{2a}\langle x-a\rangle^{2}$$

$$M(x) = -\frac{w_0}{6a}\langle x\rangle^{3} + \frac{w_0}{2}\langle x-a\rangle^{2} + \frac{w_0}{6a}\langle x-a\rangle^{3}$$

TBR 2: For the beam and loading shown and using singularity functions, express the shear and bending moment as functions of the distance *x* from the support at *A*.



$$w(x) = w_0\langle x-0\rangle^{0} - \frac{w_0}{a}\langle x\rangle^{1} + \frac{w_0}{a}\langle x-a\rangle^{1}$$

$$V(x) = -w_0\langle x\rangle^{1} + \frac{w_0}{2a}\langle x\rangle^{2} - \frac{w_0}{2a}\langle x-a\rangle^{2}$$

$$M(x) = -\frac{w_0}{2}\langle x\rangle^{2} + \frac{w_0}{6a}\langle x\rangle^{3} - \frac{w_0}{6a}\langle x-a\rangle^{3}$$