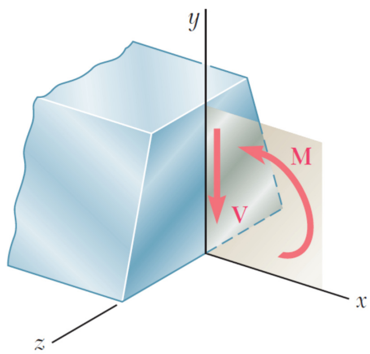
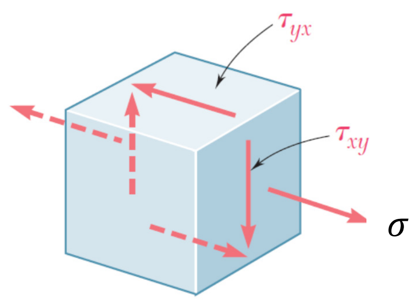
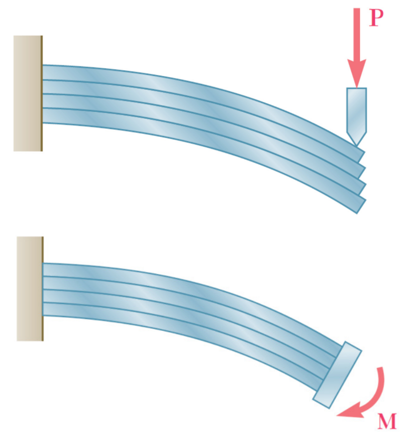
CHAPTER 6: Shearing Stresses in Beams

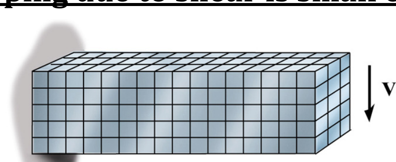
When a beam is in pure bending, the only stress resultants are the bending moments and the only stresses are the normal stresses acting on the cross sections. However, most beams are subjected to loads that produce both bending moments and shear forces. **In these cases, both normal and shear stresses are developed in the beam**. The normal stresses are calculated as explained in Chapter 4, provided the beam is constructed of a linearly elastic material. The shear stresses are discussed in this and the following two sections. The following figure expresses graphically that the elementary normal and shearing forces exerted on a given transverse section of a prismatic beam with a vertical plane of symmetry are equivalent to the bending couple **M** and the shearing force **V**.



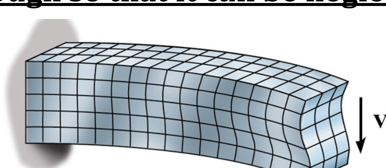




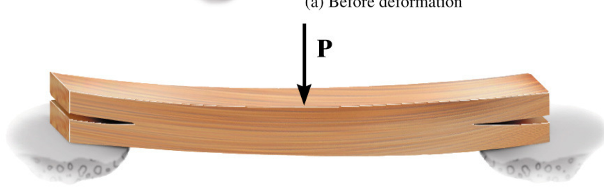
Let us now consider a small cubic element located in the vertical plane of symmetry of the beam (where we know that τ_{xz} must be zero) and examine the stresses exerted on its faces a normal stress σ_x and a shearing stress τ_{xy} are exerted on each of the two faces perpendicular to the *x* axis. But we know from Chapter 1 that, when shearing stresses τ_{xy} are exerted on the vertical faces of an element, equal stresses must be exerted on the horizontal faces of the same element. **We thus conclude that longitudinal shearing stresses must exist in any member subjected to a transverse loading** (we also conclude that shear stresses are zero at the edges: imagine this element is located at either the top or the bottom). This can be verified by considering a cantilever beam made of separate planks clamped together at one end. When a transverse load **P** is applied to the free end of this composite beam, **the planks are observed to slide with respect to each other**. In contrast, if a couple **M** is applied to the free end of the same composite beam, the various planks will bend into concentric arcs of circle and will not slide with respect to each other, thus verifying the fact that shear does not occur in a beam subjected to pure bending. As a result of shear stress, shear strain will be developed and these will tend to distort the cross section in a rather complex manner. For example, consider a short bar made of a highly deformable soft material and marked with grid lines as shown. When a shear load V is applied, it tends to deform these lines into the pattern shown and **will cause the cross section to wrap**. **Although this is the case, we can generally assume the cross sectional wrapping due to shear is small enough so that it can be neglected.**



(a) Before deformation



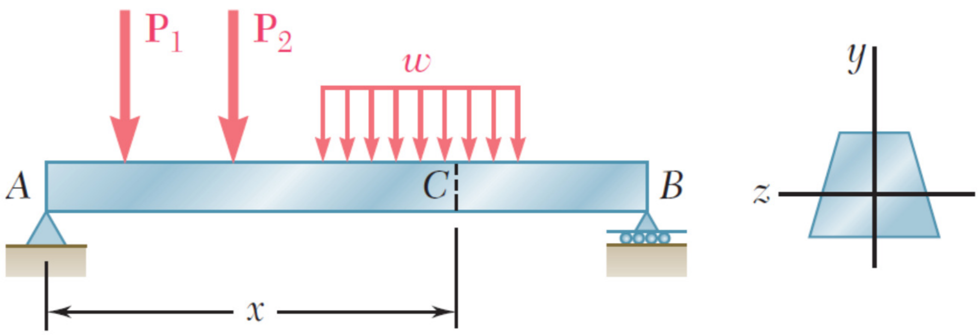
(b) After deformation

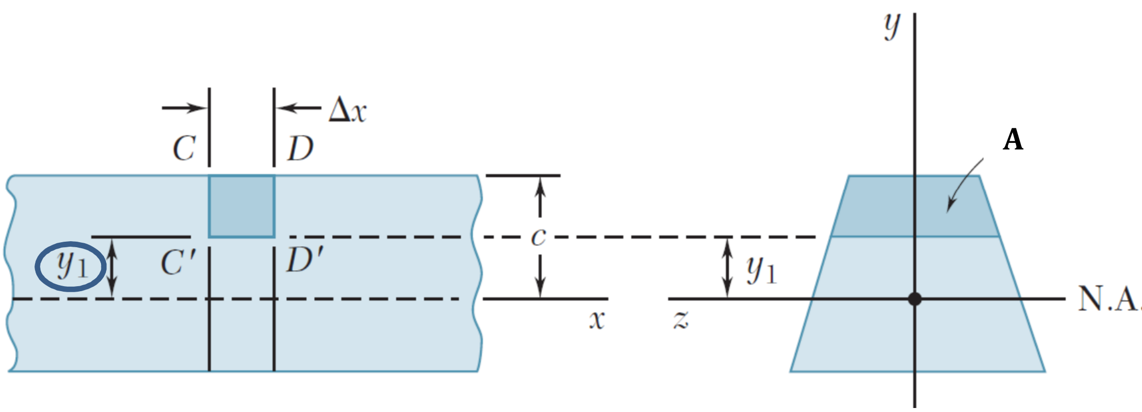


Longitudinal shear failure in timber beam

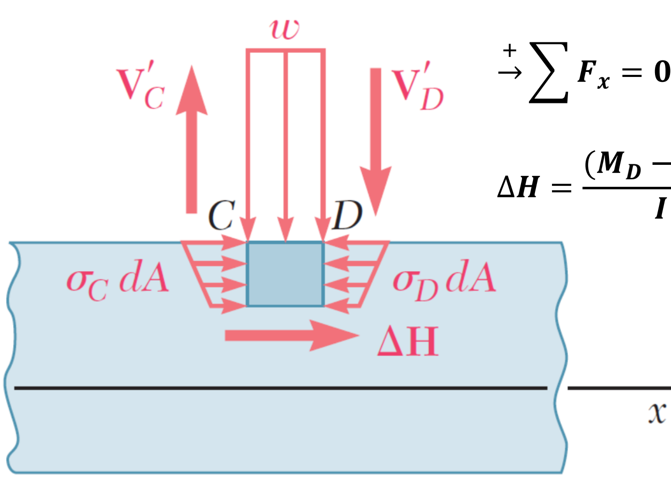
Shear on the Horizontal Face of a Beam Element

Consider a **prismatic** beam *AB* with a **vertical plane of symmetry** that supports various concentrated and distributed loads. At a distance x from end *A* we detach from the beam an element *CDD'C'* of length Δx extending across the width of the beam from the upper surface of the beam to a horizontal plane located at a distance y_1 from the neutral axis.





Now consider the forces that exert on this element:



$$\xrightarrow{+}\sum F_x = 0 \rightarrow \ \Delta H + \int_A (\sigma_C - \sigma_D)dA = 0 \ \ \text{where } \sigma = My/I$$

$$\Delta H = \frac{(M_D - M_C)}{I}\int_A y dA = \frac{\Delta M}{I}\int_A y dA = \frac{V\Delta x}{I}Q$$

$$\Delta H = \frac{VQ}{I}\Delta x \rightarrow q = \frac{VQ}{I}$$

***Q*:** first moment with respect to the neutral axis of the portion A of the cross section of the beam

***q*:** horizontal shear per unit length **(shear flow)**

$Q = A\,\frac{\int y\,dA}{A} = A\bar{Y}$: A is the area of the top (or ***bottom***) portion of the member's cross section area above (or ***below***) y_1, and $\bar{Y}$ is the distance from the neutral axis to the centroid A. The *average* *shearing stress* τ_{avg} on that face of the element is obtained by dividing ΔH by the area ΔA of the face. Observing that $\Delta A = t\,\Delta x$, where t is the width of the element at the cut, we write:

$$\boldsymbol{\tau_{avg}} = \frac{\boldsymbol{\Delta H}}{\boldsymbol{\Delta A}} = \frac{\frac{\boldsymbol{VQ}}{\boldsymbol{I}}\boldsymbol{\Delta x}}{\boldsymbol{t\,\Delta x}} = \frac{\boldsymbol{VQ}}{\boldsymbol{It}}$$

$$\boldsymbol{\tau_{avg}} = \frac{\boldsymbol{VQ}}{\boldsymbol{It}}$$

$\boldsymbol{\tau_{avg}}$ = the shear stress in the member at the point located a distance y from the neutral axis. This stress is assumed to be constant and therefore averaged across the width t of the member

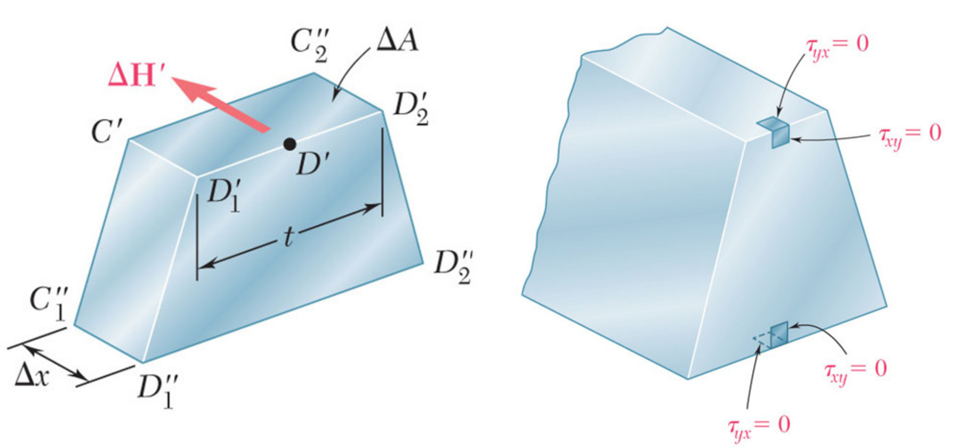
V = the internal resultant shear force, determined from the method of sections and the equations of equilibrium from “Statics”.

I = the moment of inertia of the entire cross-sectional area calculated about the neutral axis

t = the width of the member’s cross-sectional area, measured at the point where $\boldsymbol{\tau_{avg}}$ is to be determined

Q = A$\bar{Y}$, where A is the area of the top (or bottom) portion of the member’s cross-sectional area, above (or below) the section plane where $\boldsymbol{\tau_{avg}}$ is measured, and y is the distance from the neutral axis to the centroid of A.

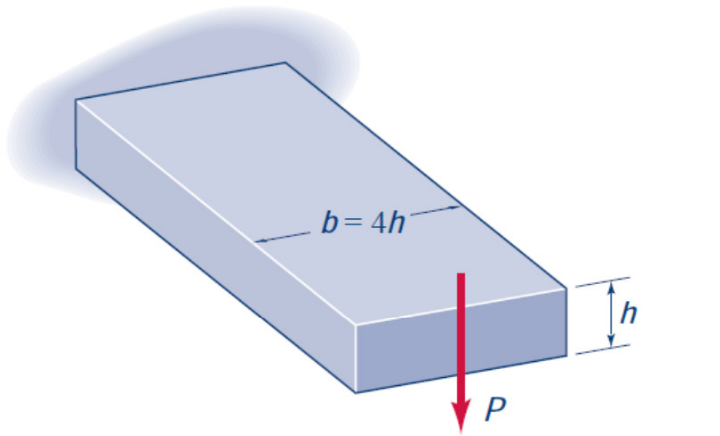
The above equation is referred to as the *shear formula* . Although in the derivation we considered only the shear stresses acting on the beam’s longitudinal plane, the formula applies as well for finding the transverse shear stress on the beam’s cross-section. Recall that these stresses are complementary and numerically equal.



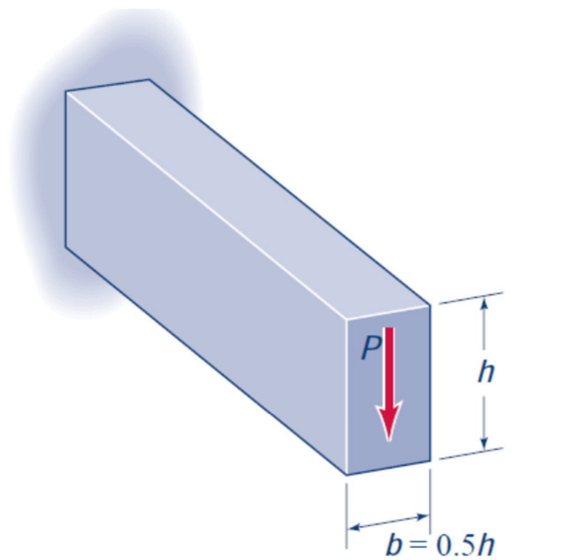
ı the lower and upper
rfaces of the beam $\tau_{yx} = 0$.
follows that $\tau_{xy} = 0$.

Limitations on the Use of the Shear Formula.

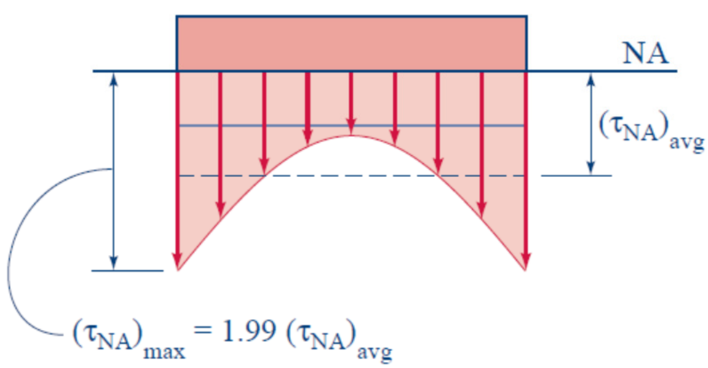
One of the major assumptions used in the development of the shear formula is that the shear stress is *uniformly* distributed over the *width* t at the section. In other words, Average value of stress ($\boldsymbol{\tau_{avg}}$) is calculated because it is assumed that shear stress remains constant across the thickness which is only true for thin sections.



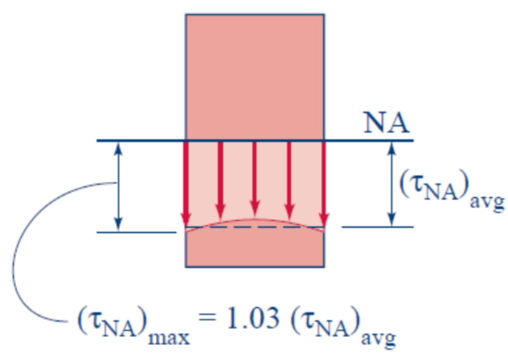
(a) A "wide beam,"



(b) A "narrow beam."

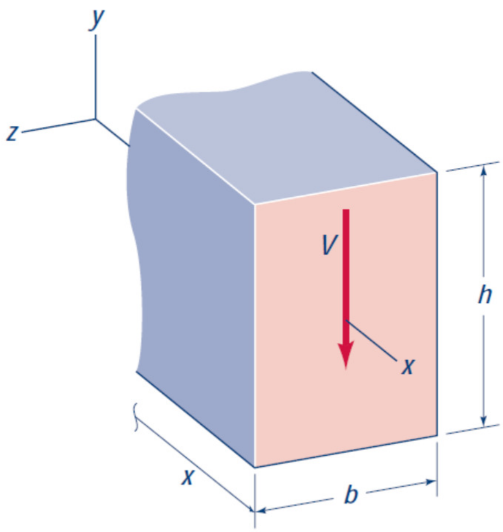


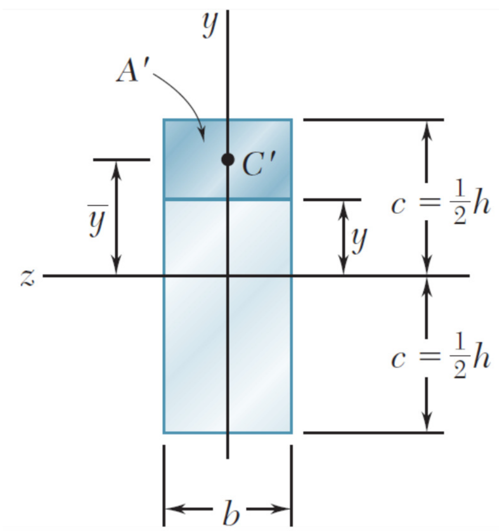
(c) Shear-stress distribution in the "wide beam"



(d) Shear-stress distribution in the "narrow beam"

Example 1: Distribution of Shear Stresses in a Rectangular Beam
The rectangular beam of width b and height h is subjected to a transverse shear force V. We aim to determine the average shear stress as a function of y, sketch the shear-stress distribution, and determine the maximum shear stress on the cross section.





$$\tau_{avg} = \frac{VQ}{It}$$

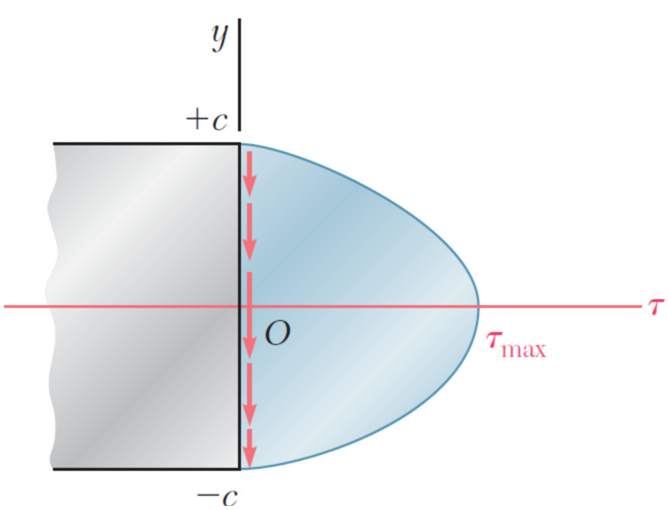
$$Q = \bar{y}A = \frac{1}{2}(y+c) \times b(c-y) = \frac{1}{2}b(c^2 - y^2) \;\; \textbf{or} \;\; Q = \int_y^c y\,dA = \int_y^c y\,b dy = b\frac{y^2}{2}\bigg|_y^c = \frac{1}{2}b(c^2-y^2)$$

$$I = \frac{1}{12}bh^3 = \frac{1}{12}b(2c)^3 = \frac{2}{3}bc^3$$

$$t = b$$

$$\tau_{avg} = \frac{VQ}{It} = \frac{V\,\frac{1}{2}b(c^2-y^2)}{\left(\frac{2}{3}bc^3\right)(b)} = \frac{3}{2}\frac{V}{2bc}\left(1 - \frac{y^2}{c^2}\right) = \frac{3}{2}\frac{V}{A}\left(1 - \frac{y^2}{c^2}\right)$$

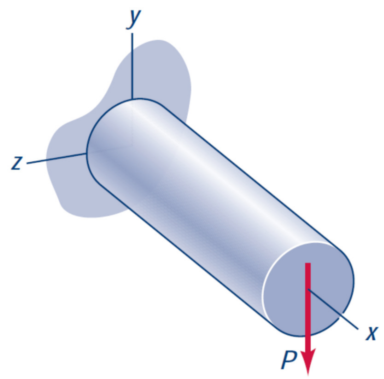
$$at\ y = 0 \rightarrow \tau_{max} = \frac{3}{2}\frac{V}{A} \qquad and \qquad at\ y = \pm c \rightarrow \tau = 0$$

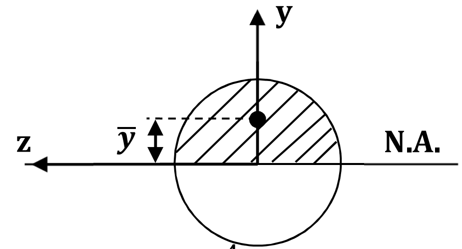


The relation obtained shows that the maximum value of the shearing stress in a beam of rectangular cross section is 50% larger than the value V/A that would be obtained by wrongly assuming (as in Chapter 1) a uniform stress distribution across the entire cross section.

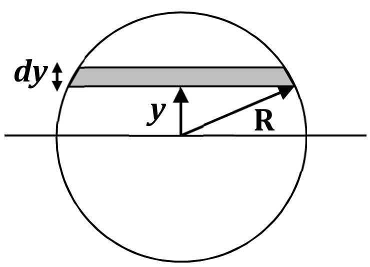
Example 2: Distribution of Shear Stresses in a Circular Beam

The circular beam of radius r is subjected to a transverse shear force V. We aim to determine the maximum shear stress on the cross section.





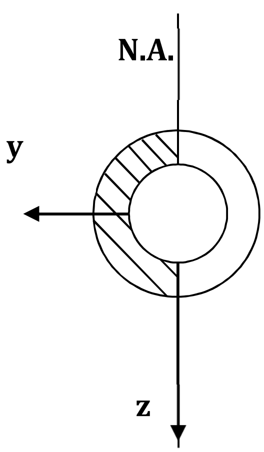
$$\tau_{max} = \frac{VQ}{It} = \frac{V(\frac{4r}{3\pi} \times \frac{\pi}{2} r^2)}{(\frac{1}{4}\pi r^4)(2r)} = \frac{4}{3}\frac{V}{\pi r^2} = \frac{4}{3}\frac{V}{A}$$

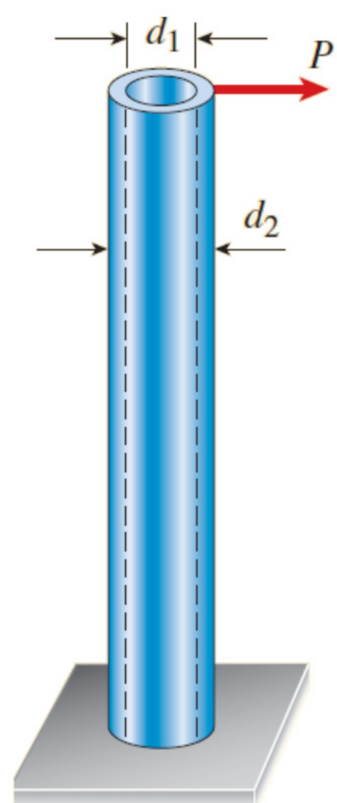


$$Q = \int y\, dA = \int y\, \left(2\sqrt{R^2 - y^2}\, dy\right) \rightarrow$$

$$Q_{max} = \int_0^R 2y\,\sqrt{R^2 - y^2}\, dy = \frac{2}{3} R^3$$

Example 3: Determine the maximum shear stress in the beam with a hollow circular cross section as shown.





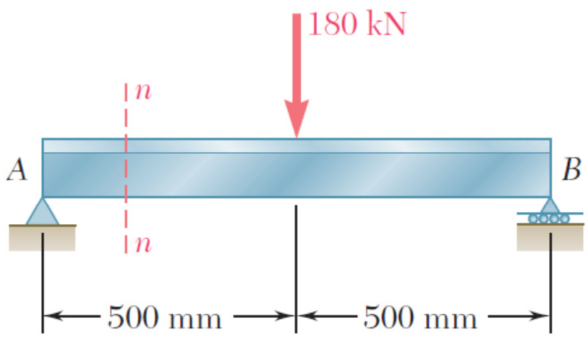
$$\tau_{max} = \frac{VQ}{It}$$

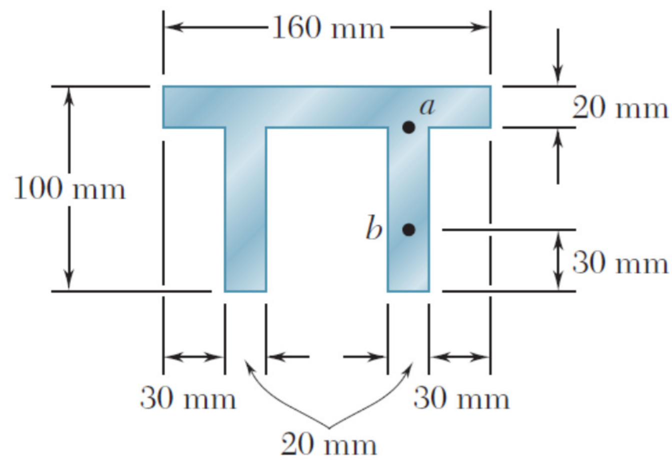
$$Q = \frac{4r_2}{3\pi} \times \frac{\pi}{2} r_2^2 - \frac{4r_1}{3\pi} \times \frac{\pi}{2} r_1^2 = \frac{2}{3}(r_2^3 - r_1^3)$$

$$I = \frac{1}{4}\pi r_2^4 - \frac{1}{4}\pi r_1^4 = \frac{\pi}{4}(r_2^4 - r_1^4), t = 2(r_2 - r_1)$$

$$\tau_{max} = \frac{VQ}{It} = \frac{P \times \frac{2}{3}(r_2^3 - r_1^3)}{\frac{\pi}{4}(r_2^4 - r_1^4) \times 2(r_2 - r_1)} = \frac{4P}{3\pi}\left(\frac{r_2^2 + r_1 r_2 + r_1^2}{r_2^4 - r_1^4}\right)$$

Example 4: For the beam and loading shown, consider section *n-n* and determine the shearing stress at (*a*) point *a,* (*b*) point *b,* and (*c*) the largest shearing stress at section *n-n.*





$$\tau = \frac{VQ}{It}$$

From Statics: $V = 90\ kN,\ \ \bar{y} = \frac{\sum yA}{\sum A} = \frac{90\times(20\times160)+2\times[40\times(80\times20)]}{20\times160+2\times80\times20} = 65\ mm$

$(position\ of\ N.A.\ with\ respect\ to\ the\ lower\ end)$

$$I_{N.A.} = \frac{1}{12}160\times20^3 + 160\times20\times(90-65)^2 + 2\times\left[\frac{1}{12}20\times80^3 + 80\times20\times(65-40)^2\right]$$
$$= 5.81\times10^6\ mm^4$$

Part (a): $Q_a = \bar{y}A = (90-65)\times20\times160 = 80\ 000\ mm^3$

$$\tau_a = \frac{VQ_a}{It_a} = \frac{(90\ 000\ N)(80\ 000\ mm^3)}{(5.81\times10^6\ mm^4)(2\times20\ mm)} = 30.98\ MPa$$

Or: $Q_a = \bar{y}A = -(65-40)\times20\times80 = -40\ 000\ mm^3, \tau_a = \frac{VQ_a}{It_a} = \frac{(90\ 000\ N)(-40\ 000\ mm^3)}{(5.81\times10^6\ mm^4)(20\ mm)} = -30.98\ MPa$

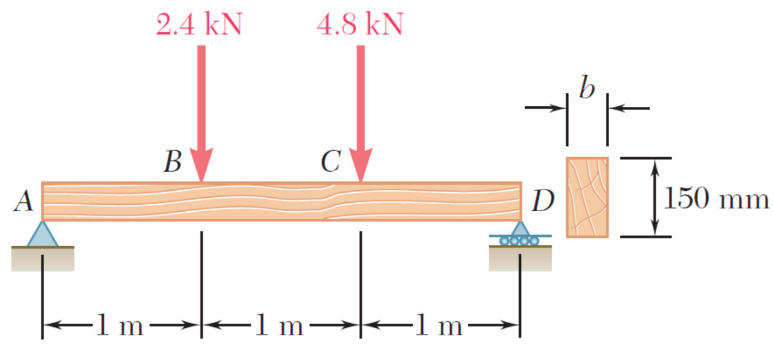
Part (b): $Q_b = \bar{y}A = -(65-15)\times20\times30 = -30\ 000\ mm^3$,

$$\tau_b = \frac{VQ_b}{It_b} = \frac{(90\ 000\ N)(-30\ 000\ mm^3)}{(5.81\times10^6\ mm^4)(20\ mm)} = -23.23\ MPa$$

Part (c): $Q_{NA} = \bar{y}A = 2\times\left(\frac{65}{2}\times65\times20\right) = 84\ 500\ mm^3$,

$$\tau_{max} = \frac{VQ_b}{It_b} = \frac{(90\ 000\ N)(84\ 500\ mm^3)}{(5.81\times10^6\ mm^4)(2\times20\ mm)} = 32.72\ MPa$$

Example 5: For the beam and loading shown, determine the minimum required width b, knowing that for the grade of timber used, σ_{all} = 12 MPa and τ_{all} = 825 kPa.



From Statics: R_A = 3.2 kN, R_D = 4 kN

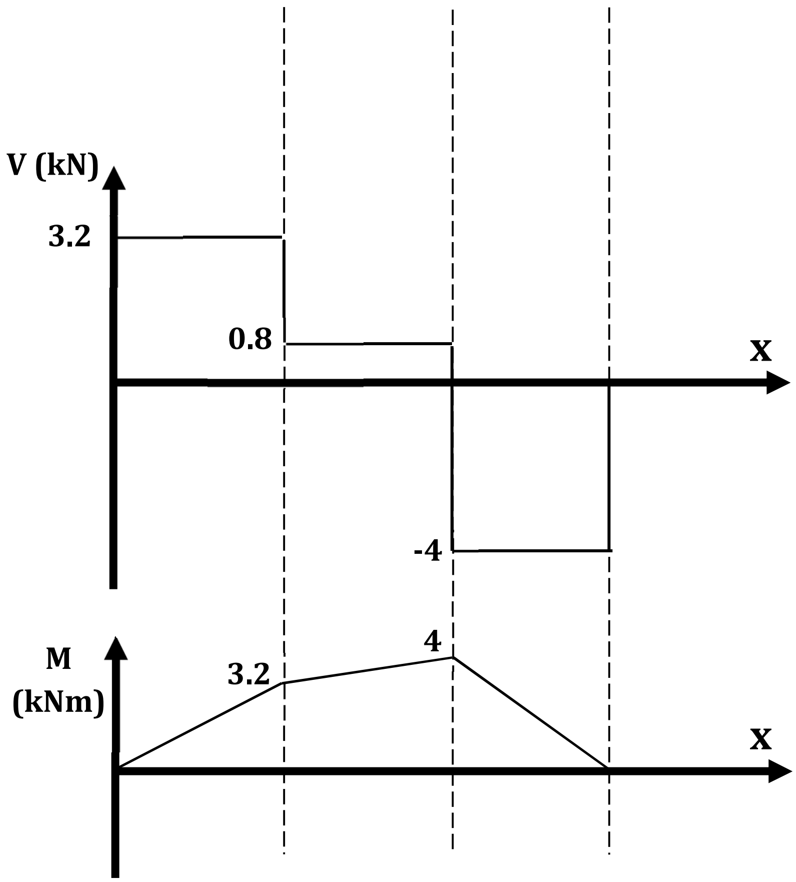
Critical point is C:

M_{max} = 4 kNm, V_{max} = -4 kN

$$\sigma_{max} = \frac{M_{max}c}{I} \rightarrow$$

$$12\ MPa = \frac{(4 \times 10^6\ Nmm)(75\ mm)}{\frac{1}{12} b \times 150^3\ mm^4}$$

$$\rightarrow b = 88.9\ mm$$



$$\tau_{max} = \frac{VQ}{It})_{max} \rightarrow$$

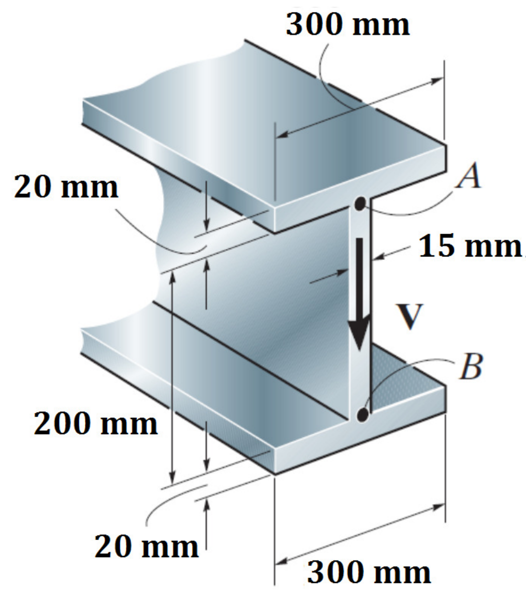
$$0.825\ MPa = \frac{(4000\ N)\left(\frac{75}{2} \times b \times 75\ mm^3\right)}{(\frac{1}{12} b \times 150^3\ mm^4)b}$$

$$\rightarrow b = 48.5\ mm$$

Answer:

$$\rightarrow b = 88.9\ mm$$

Example 6: A beam of wide-flange shape is subjected to a vertical shear force V = 80 kN. The cross-sectional dimensions of the beam are shown. Determine the distribution of shear stress over the cross section of the beam.



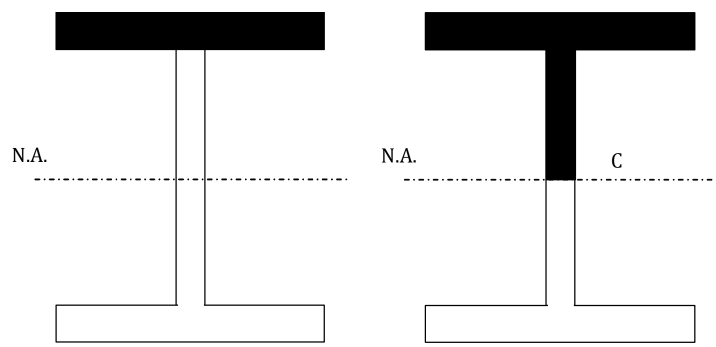
At upper surface of the cross section:

$$\tau = 0$$

At point A on the flange:

$$\tau = \frac{VQ}{It}$$

$$I = \frac{1}{12}15 \times 200^3 + 2 \times \left\{\frac{1}{12}300 \times 20^3 + 20 \times 300 \times 110^2\right\} = 155.6 \times 10^6\ mm^4$$



$$Q = A\bar{y} = 20 \times 300 \times 110 = 660000\ mm^3$$

$$t = 300\ mm$$

$$\tau = \frac{VQ}{It} = \frac{(80\,000\ N)(660000\ mm^3)}{(155.6 \times 10^6\ mm^4)(300\ mm)} = 1.13\ MPa$$

At point A on the web:

$$t = 15\ mm$$

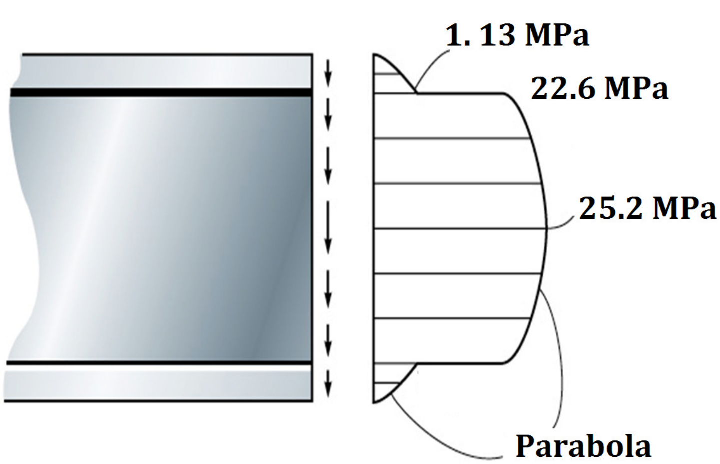
$$\tau = \frac{VQ}{It} = \frac{(80\,000\ N)(660000\ mm^3)}{(155.6 \times 10^6\ mm^4)(15\ mm)} = 22.62\ MPa$$

At point C on the mid-web:

$$Q = A\bar{y} = 20 \times 300 \times 110 + 15 \times 100 \times 50 = 735000\ mm^3$$

$$t = 15\ mm$$

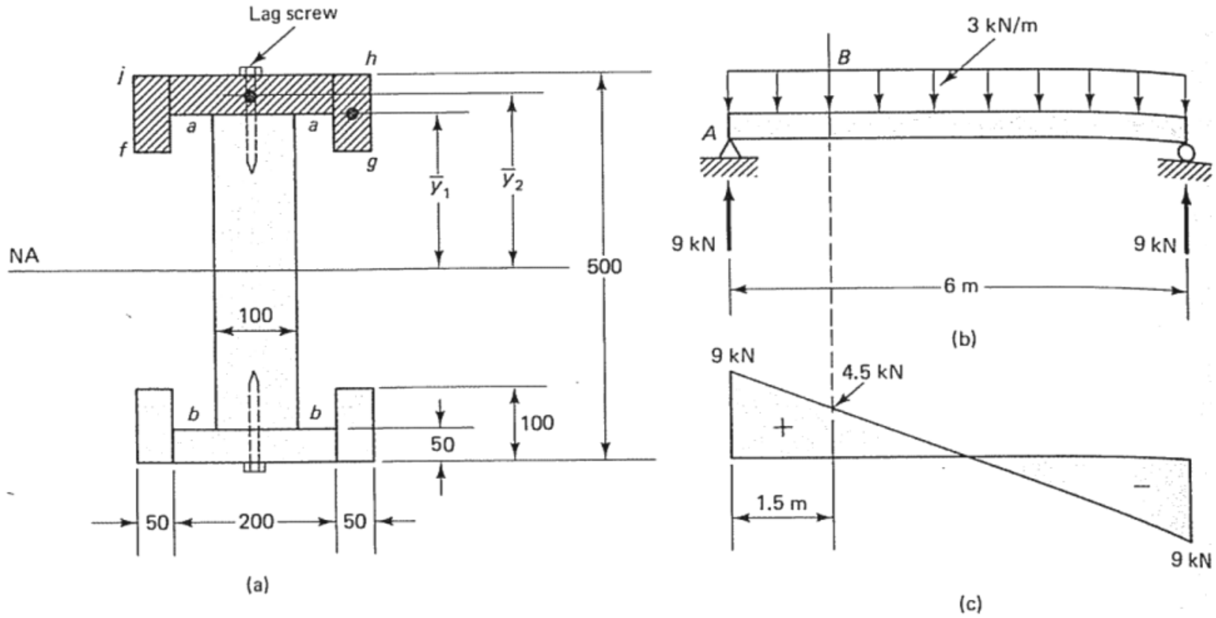
$$\tau = \frac{VQ}{It} = \frac{(80\,000\ N)(735000\ mm^3)}{(155.6 \times 10^6\ mm^4)(15\ mm)} = 25.2\ MPa$$



Note that most of the shear stress occurs in the web and is almost uniform throughout its depth, varying from 22.6 MPa to 25.2 MPa. It is for this reason that in practice, one usually assumes that the entire shear load is carried by the web, and that a good approximation of the maximum value of the shearing stress in the cross section can be obtained by dividing V by the cross-sectional area of the web:

$$\tau_{max} = \frac{V}{A_{web}} = \frac{80\,000\ N}{15 \times \mathbf{240\ mm^2}} = 22.2\ MPa$$

Example 7: For the beam and loading shown, each screw is good for supporting 2 kN lateral load to avoid longitudinal sliding at *a*-*a*, determine the minimum required distance between screws.



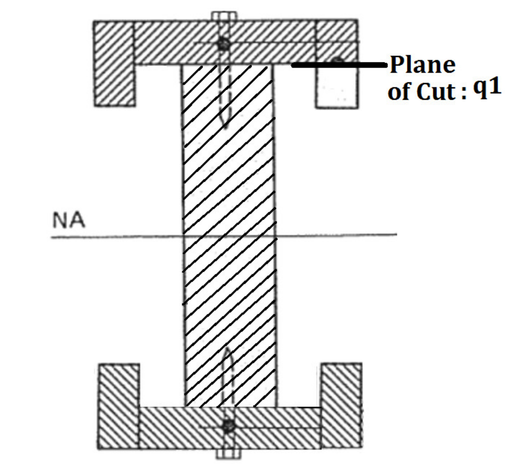
$V_{max} = 9\ kN\ and\ Q_{a-a} = \sum \bar{y}A = 225 \times (200 \times 50) + 200 \times (50 \times 100) + 200 \times (50 \times 100) = 4\ 250\ 000\ mm^3,\ I = 2.36 \times 10^9\ mm^4 \rightarrow q = \frac{VQ}{I} = \frac{(9000\ N)(4\ 250\ 000\ mm^3)}{2.36 \times 10^9\ mm^4} = 16.2\ N/mm$

Spacing of screws: S = (1 mm)(2000 N)/16.2 N =123.45 mm

1 mm 16.2 N

S 2000 N

S = 120 mm is okay



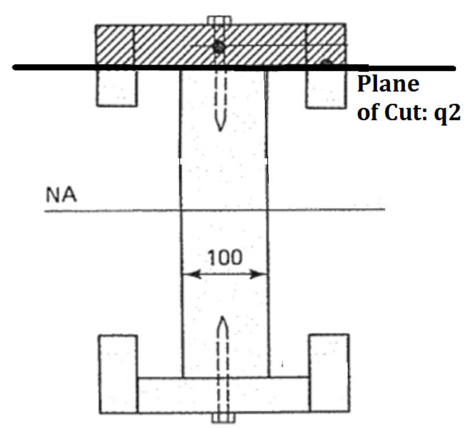
------------------------ **CHECKING** ------------------------

$$Q_1 = \sum \bar{y}A = 225 \times (300 \times 50) + 175 \times (50 \times 50) - 225 \times (300 \times 50) - 175 \times (50 \times 50) \times 2 = -437\ 500\ mm^3$$

OR: $Q_1 = \sum \bar{y}A = -175 \times 50 \times 50 = -437\ 500\ mm^3$

$$q_1 = \frac{VQ_1}{I} = \frac{(9000\ N)(-437\ 500\ mm^3)}{2.36 \times 10^9\ mm^4} = -1.668\ N/mm$$

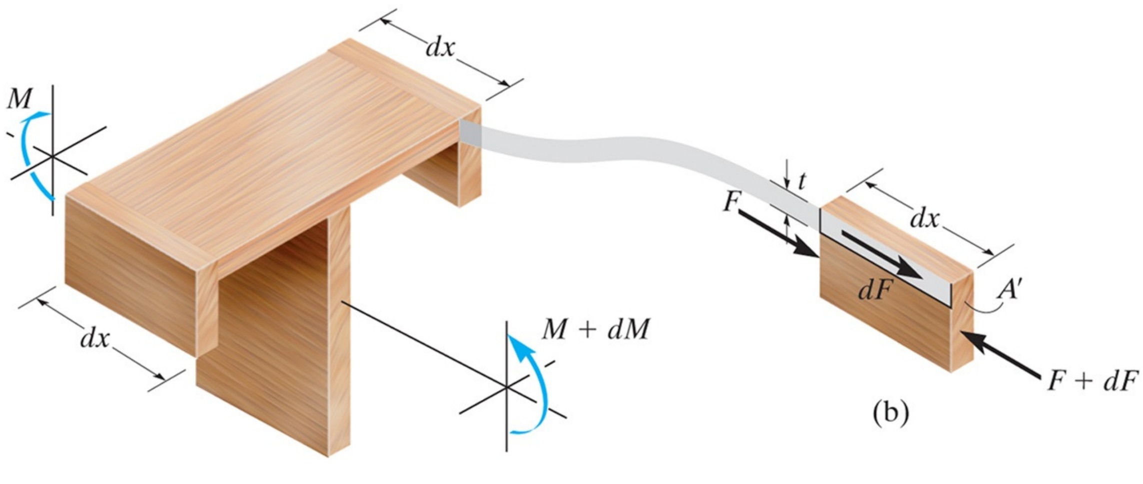
$$Q_2 = \sum \bar{y}A = 225 \times (300 \times 50) = 3\ 375\ 000\ mm^3$$



$$q_2 = \frac{VQ_1}{I} = \frac{(9000\ N)(3\ 375\ 000\ mm^3)}{2.36 \times 10^9\ mm^4} = 12.870\ N/mm$$

$check: q_2 = q + 2q_1 \rightarrow 12.87 = 16.2 + 2(-1.668)$ ✓

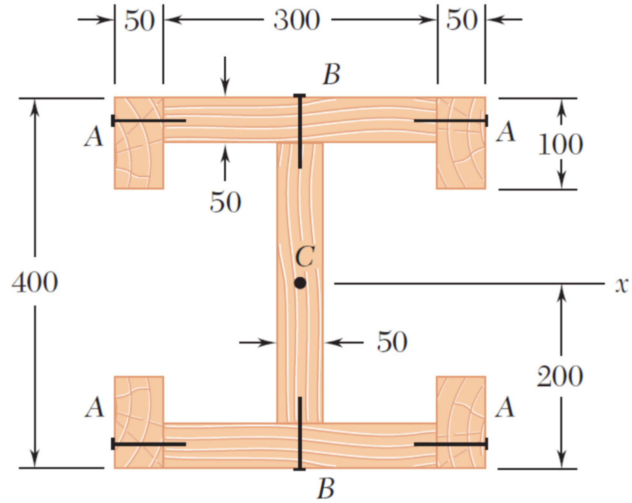
Shear in the Longitudinal Direction of a Beam Element



(a)

$$\overset{+}{\rightarrow}\sum \boldsymbol{F_x} = \mathbf{0} \rightarrow\ \Delta \boldsymbol{H} + \int_{\boldsymbol{A}} (\boldsymbol{\sigma_C} - \boldsymbol{\sigma_D})\boldsymbol{dA} = \mathbf{0}\ \ \boldsymbol{where}\ \boldsymbol{\sigma} = \boldsymbol{My/I}$$

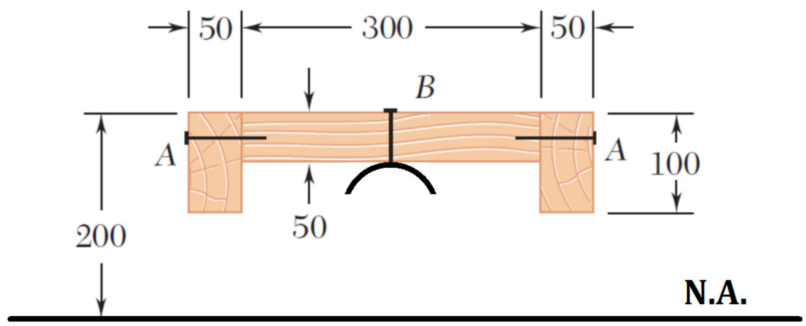
$$\Delta \boldsymbol{H} = \frac{(\boldsymbol{M_D} - \boldsymbol{M_C})}{\boldsymbol{I}}\int_{\boldsymbol{A}} \boldsymbol{ydA} = \frac{\Delta \boldsymbol{M}}{\boldsymbol{I}}\int_{\boldsymbol{A}} \boldsymbol{ydA} = \frac{\boldsymbol{V}\Delta \boldsymbol{x}}{\boldsymbol{I}}\boldsymbol{Q} \rightarrow \Delta \boldsymbol{H} = \frac{\boldsymbol{VQ}}{\boldsymbol{I}}\Delta \boldsymbol{x} \rightarrow \boldsymbol{q} = \frac{\boldsymbol{VQ}}{\boldsymbol{I}} \rightarrow\ \boldsymbol{\tau_{avg}} = \frac{\Delta \boldsymbol{H}}{\Delta \boldsymbol{A}} = \frac{\frac{\boldsymbol{VQ}}{\boldsymbol{I}}\Delta \boldsymbol{x}}{\boldsymbol{t}\,\Delta \boldsymbol{x}} = \frac{\boldsymbol{VQ}}{\boldsymbol{It}}$$



Example 8: The built-up wooden beam shown is subjected to a vertical shear of 8 kN. Knowing that the nails are spaced longitudinally every 60 mm at *A* and every 25 mm at *B*, determine the shearing force in the nails (*a*) at *A*, (*b*) at *B* (*Given:* I_x = 1.504 × 10^9 mm^4). Dimensions are in mm.

$q = \dfrac{VQ}{I}, V = 8000\ N, I = 1.504 \times 10^9 mm^4,$

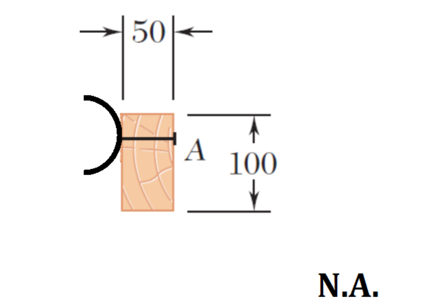
$Q = ?$



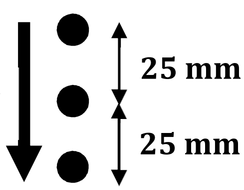
At B:

$$Q = \sum \bar{y}A = (200 - 25)(400 \times 50) + 2(200 - 75) \times (50 \times 50) = 4\,125\,000\ mm^3$$

$$q = \frac{VQ}{I} = \frac{(8000\ N)(4\,125\,000\ mm^3)}{1.504 \times 10^9 mm^4}$$



$\boldsymbol{q = 21.94\ N/mm}$



$$F = q \times S = 21.94\ N/mm \times 25\ mm = 549\ N$$

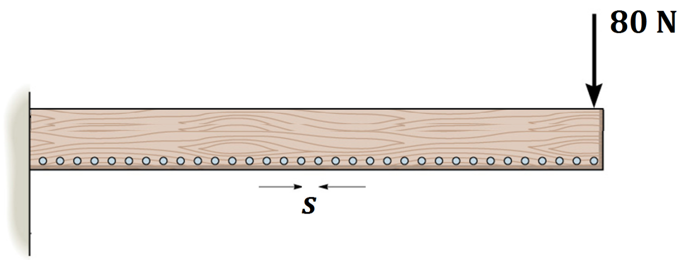
At A:

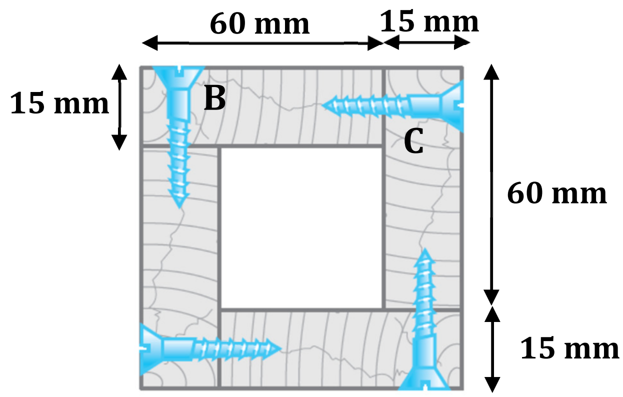
$$Q = \sum \bar{y}A = (200 - 50)(100 \times 50) = 750\,000\ mm^3$$

$$q = \frac{VQ}{I} = \frac{(8000\ N)(750\,000\ mm^3)}{1.504 \times 10^9 mm^4} = 3.99\ N/mm$$

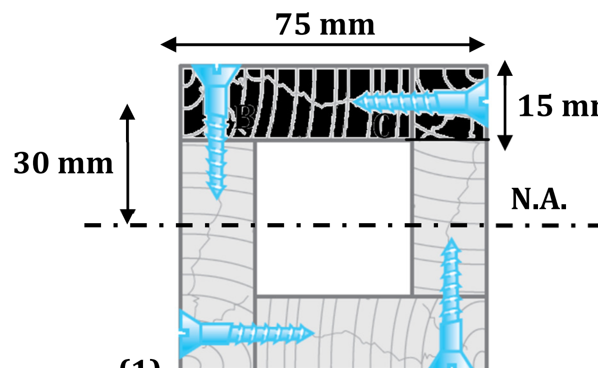
$$F = q \times S = 3.99\ N/mm \times 60\ mm = 239.3\ N$$

Example 9: A box beam is constructed from four boards nailed together as shown. If each nail can support a shear force of 30 N, determine the maximum spacing ($\boldsymbol{s}$) of the nails at B and at C that beam will support the force of 80 N.





$$I_{NA} = \frac{1}{12}75(75^3) - \frac{1}{12}45(45^3) = 2\ 295\ 000\ mm^4$$



(1)

$$Q_1 = \bar{y}A = (30)(75 \times 15) = 33\ 750\ mm^3$$

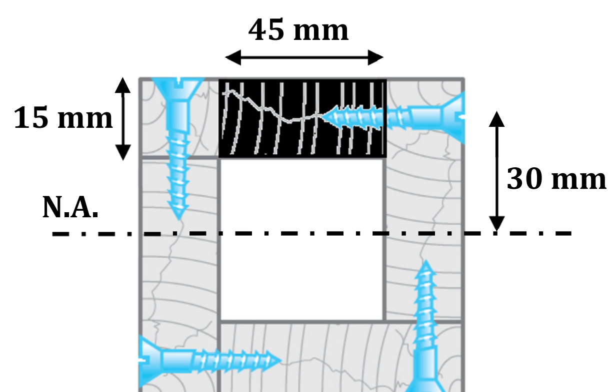
$$q_1 = \frac{1}{2}\frac{VQ_1}{I} = \frac{1}{2}\frac{(80\ N)(33\ 750\ mm^3)}{2\ 295\ 000\ mm^4} = 0.588\ N/mm$$

$$s_1 = \frac{30\ N}{0.588\ N/mm} = 51\ mm$$

$$Q_2 = \bar{y}A = (30)(45 \times 15) = 20\ 250\ mm^3$$

$$q_2 = \frac{1}{2}\frac{VQ_2}{I} = \frac{1}{2}\frac{(80\ N)(20\ 250\ mm^3)}{2\ 295\ 000\ mm^4} = 0.353\ N/mm$$

$$s_1 = \frac{30\ N}{0.353\ N/mm} = 85\ mm$$



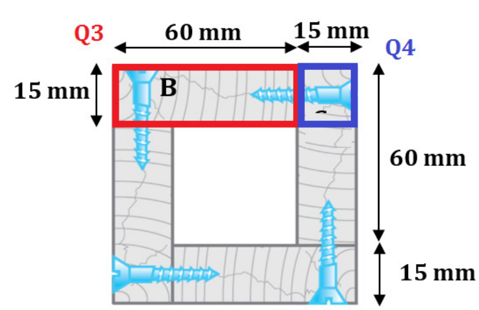
(2)

OR:

$$Q_3 = \bar{y}A = (30)(60 \times 15) = 27\ 000\ mm^3$$

$$q_1 + q_2 = \frac{VQ_3}{I} = \frac{(80\ N)(27\ 000\ mm^3)}{2\ 295\ 000\ mm^4} = 0.941 N/mm$$

We know: $q_1 = 0.588\ N/mm \rightarrow q_2 = 0.941 - 0.588 = 0.353\ N/mm$



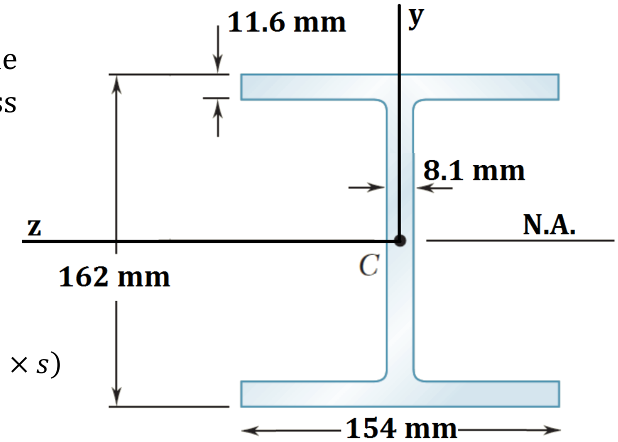
OR:

$$Q_4 = \bar{y}A = (30)(15 \times 15) = 6\ 750\ mm^3$$

$$q_1 + q_2 = \frac{VQ_4}{I} = \frac{(80\ N)(6\ 750\ mm^3)}{2\ 295\ 000\ mm^4} = 0.235\ N/mm$$

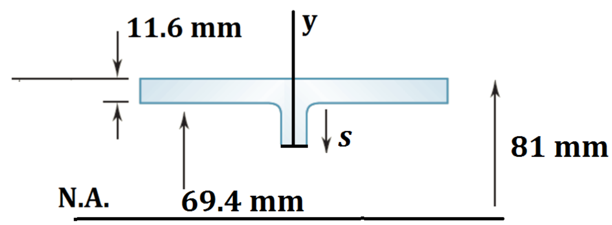
We know: $q_1 = 0.588\frac{N}{mm} \rightarrow q_2 = 0.235 - 0.588 = -0.353\ N/mm$

Example 10: Determine the shear stress distribution in the flanges and the web of the thin-walled beam with the cross section shown subjected to the shear force $V = 60$ kN.



$I_{N.A.} = 22.2 \times 10^6 \; mm^4$

For the web:



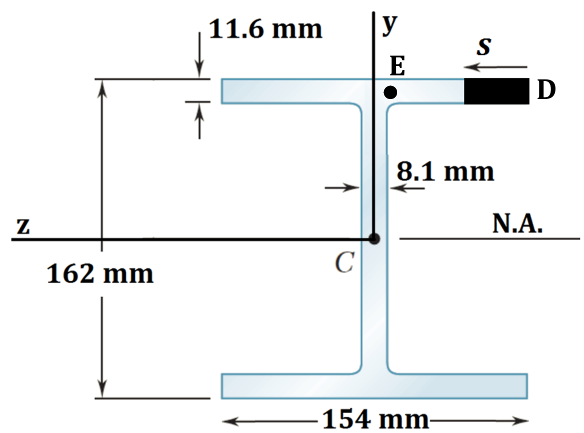
$$Q = \sum \bar{y}A = \left(81 - \frac{11.6}{2}\right)(11.6 \times 154) + \left(69.4 - \frac{s}{2}\right)(8.1 \times s)$$
$$= 134337.3 + 562.14\, s - 4.05\, s^2$$

$$\tau_{web} = \frac{VQ}{It} = \frac{(60\,000\; N)(134337.3 + 562.14\, s - 4.05\, s^2)\; mm^3}{(22.2 \times 10^6\; mm^4)(8.1\; mm)}$$

$at\ s = 0\ (upper\ end\ of\ the\ web\ at\ point\ B) \rightarrow \tau = 44.8\ MPa$

$at\ s = 69.4\ mm\ (mid - web\ at\ point\ C) \rightarrow \tau = 51.3\ MPa$

For the flanges:



$$Q = \bar{y}A = \left(81 - \frac{11.6}{2}\right)(11.6\, s) = 872.32\, s$$

$$\tau_{flange} = \frac{VQ}{It} = \frac{(60\,000\; N)(872.32\, s)\; mm^3}{(22.2 \times 10^6\; mm^4)(11.6\; mm)}$$

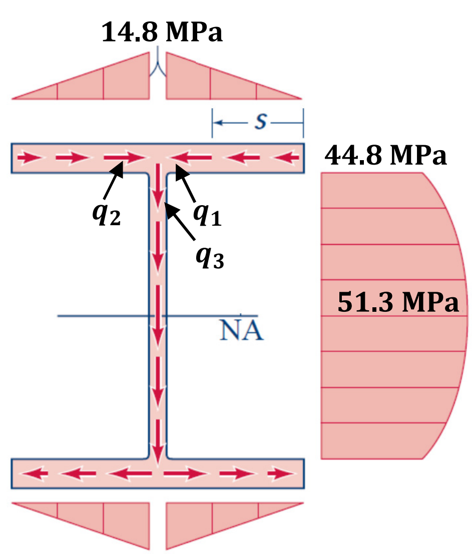
$at\ s = 0\ (point\ D) \rightarrow \tau = 0\ MPa$

$at\ s = 72.95\ mm\ (point\ E) \rightarrow \tau = 14.8\ MPa$

Calculating shear flow:

$$q_1 = \frac{VQ}{I} = \tau t = \frac{(60\,000\; N)(872.32\, s)\; mm^3}{(22.2 \times 10^6\; mm^4)} = 2.35\, s$$

$$At\ mid\ flange\ \left(s = \frac{154}{2} mm\right):\ q_1 = q_2 = 181.5\ N/mm$$



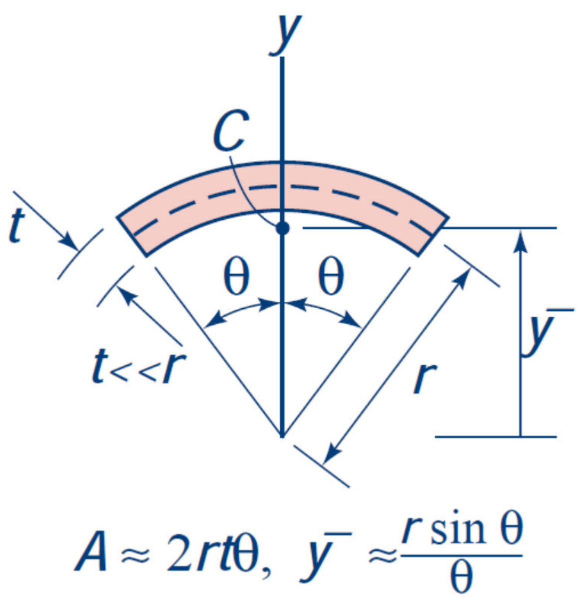
$$q_3 = \frac{VQ}{I} = \tau t = \frac{(60\,000\; N)(134337.3 + 562.14\, s - 4.05\, s^2)\; mm^3}{(22.2 \times 10^6\; mm^4)}$$

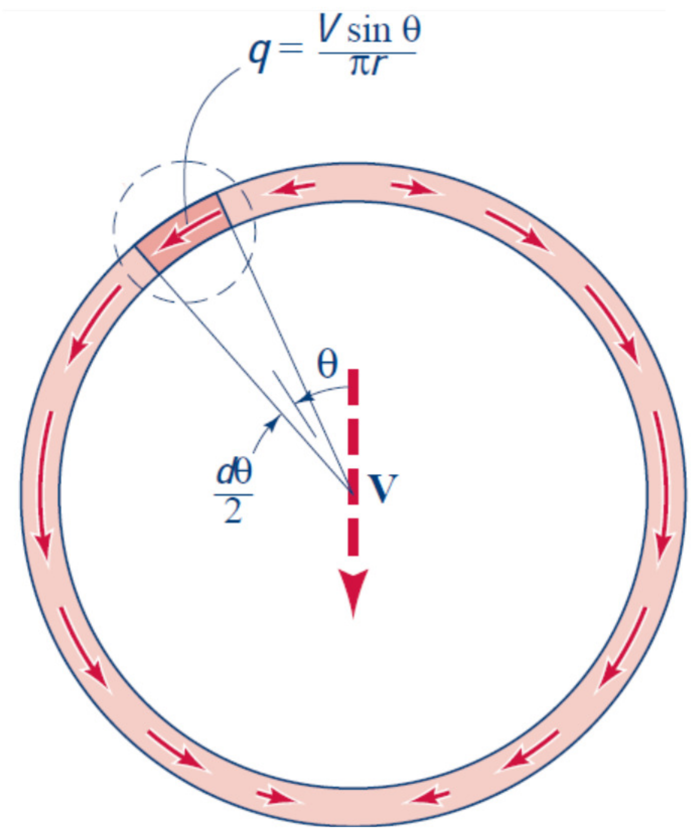
$at\ s = 0\ (upper\ end\ of\ the\ web\ at\ point\ B) \rightarrow q_3 = 363\ N/mm$

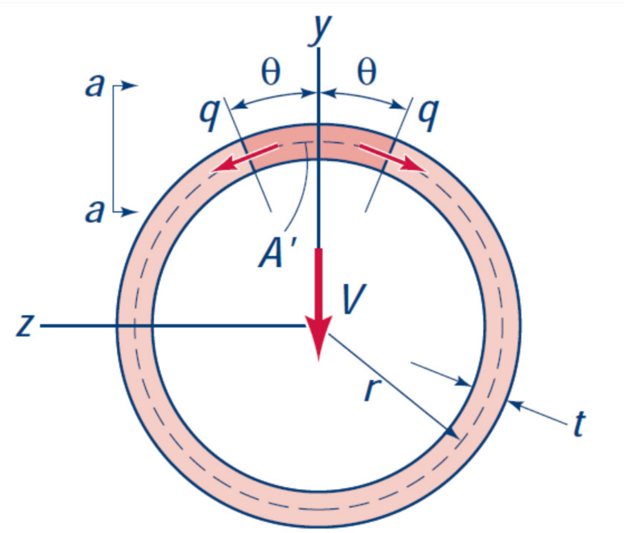
Note that at B we have: $q_1 + q_2 = q_3$

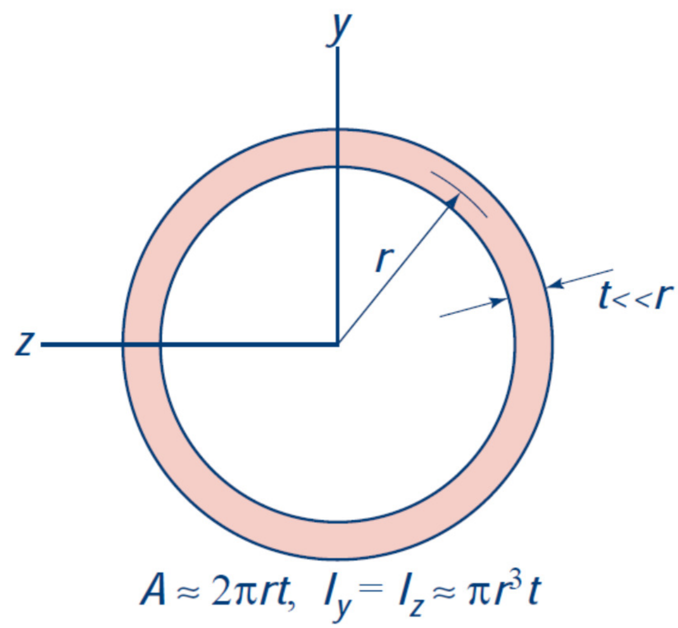
1- **τ is linear in the flanges and parabolic in the web**
2- **τ and q are parallel to the walls**
3- **τ is equal to zero at free surfaces**
4- **VQ/It holds true only where we have thin wall**
5- **Equilibrium of forces and moments is satisfied**
6- **Shear flow is similar to fluid flow, i.e., $q_1 + q_2 = q_3$**

TBR 1: Determine the shear flow distribution in the beam with the cross section shown subjected to a shear force of V.



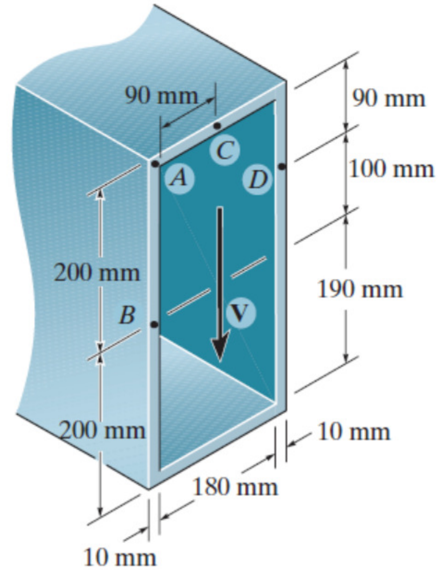






(145)

TBR 2: A shear force of 450 N is applied to the box girder. Determine the shear flow at points C and D.



$$I = \frac{1}{12}(0.2)(0.4^3) - \frac{1}{12}(0.18)(0.38^3) = 0.24359(10^{-3})\text{ m}^4$$

Referring to Fig. *a*, due to symmetry $A'_C = 0$. Thus

$$Q_C = 0$$

Then referring to Fig. *b*,

$$Q_D = \bar{y}'_1 A'_1 + \bar{y}'_2 A'_2 = 0.195(0.01)(0.09) + 0.15(0.1)(0.01)$$

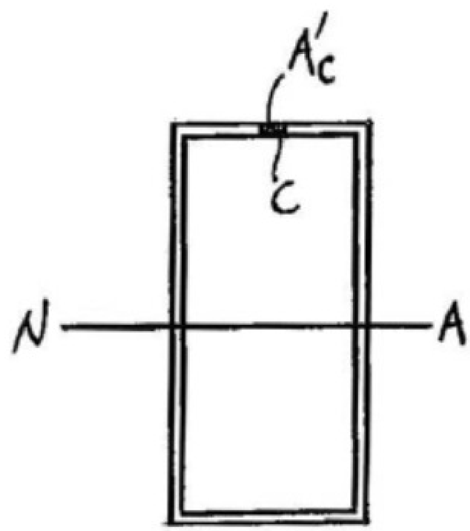
$$= 0.3255(10^{-3})\text{ m}^3$$

Thus,

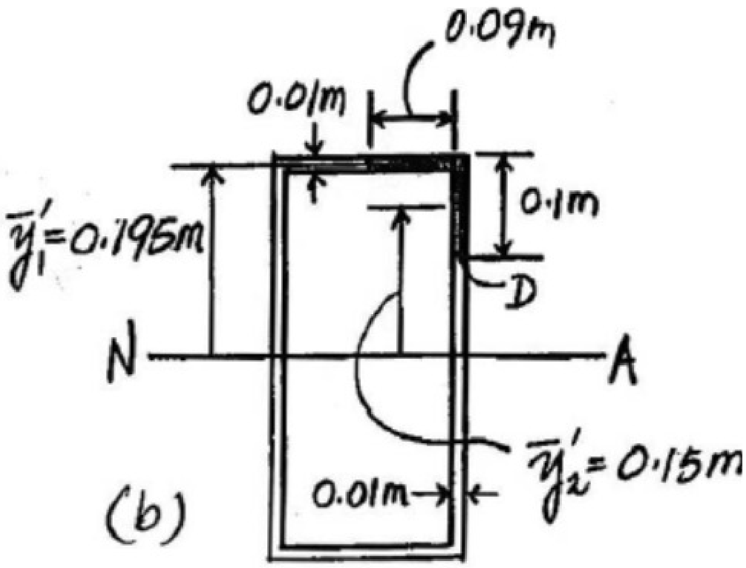
$$q_C = \frac{VQ_C}{I} = 0$$

$$q_D = \frac{VQ_D}{I} = \frac{450(10^3)\left[0.3255(10^{-3})\right]}{0.24359(10^{-3})}$$

$$= 601.33(10^3)\text{ N/m} = 601\text{ kN/m}$$

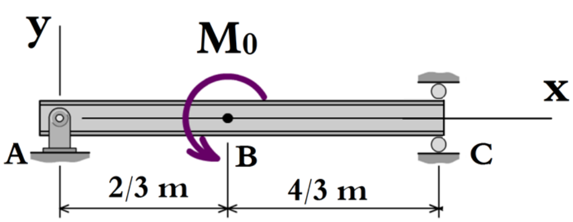


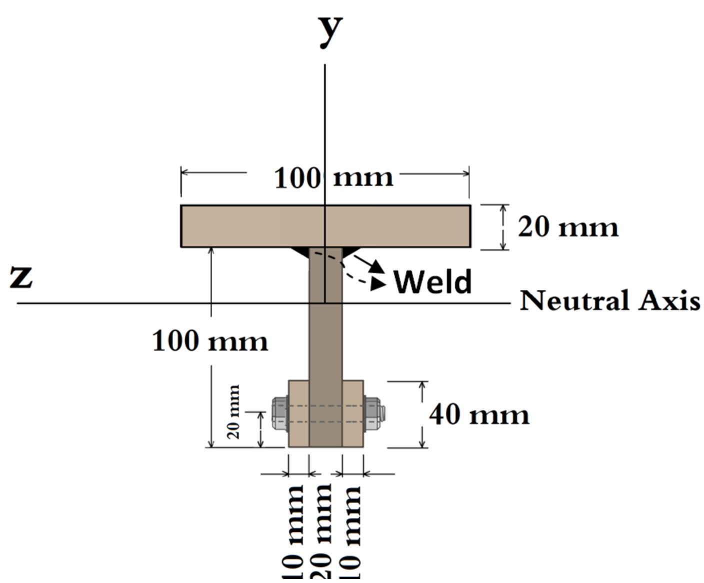
(a)

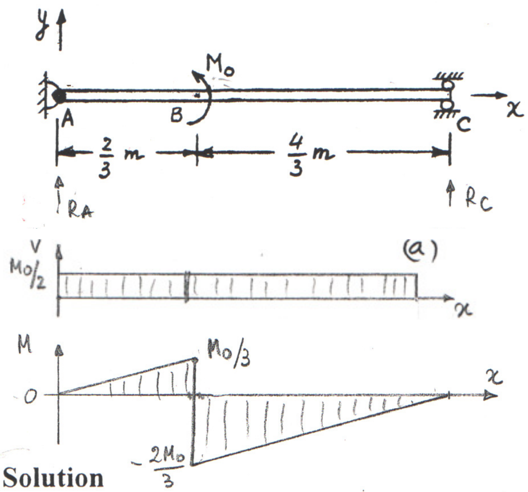


(b)

TBR 3: Screws are placed at distance of 250 mm throughout the length of the beam and their diameter is 6 mm. The welding is done throughout the length. Find maximal value of M_0 so that (1) the maximal tensile bending stress in the beam remains smaller than 120 MPa, (2) shear flow (q) in the weld remains smaller than 38 kN/m, and (c) shear stress in each screw remains smaller than 90 MPa (1390).







Solution

$\Sigma M_{z/A} = 0 \rightarrow R_C = -M_0/2$

$\Sigma F_y = 0 \rightarrow R_A = M_0/2$

$$\bar{Y} = \frac{(100 \times 20 \times 110) + (100 \times 20 \times 50) + 2(10 \times 40 \times 20)}{(100 \times 20) + (100 \times 20) + 2(10 \times 40)} = 70 \text{ mm}$$

$I_z = I_1 + I_2 + I_3$

$I_1 = \frac{1}{12} \times 100 \times 20^3 + (50-10)^2 \times 100 \times 20 = 3{,}266 \times 10^6 \text{ mm}^4$

$I_2 = \frac{1}{12} \times 20 \times 100^3 + (70-50)^2 \times 100 \times 20 = 2{,}466 \times 10^6 \text{ mm}^4$

$I_3 = 2\left[\frac{1}{12} \times 10 \times 40^3 + (70-20)^2 \times 10 \times 40\right] = 2{,}1067 \times 10^6$

$I_z = 7{,}840 \times 10^6 \text{ mm}^4 = 7{,}840 \times 10^{-6} \text{ m}^4$

a)

$\sigma_{B^-} = \frac{M_0}{3} \times \frac{0{,}070}{I_z} = \frac{M_0}{I} \times 2{,}333 \times 10^{-2}$

$\sigma_{B^+} = \frac{2M_0}{3} \times \frac{0{,}050}{I_z} = \frac{M_0}{I} \times 3{,}33 \times 10^{-2} \leftarrow \sigma_t^{MAX}$

Ce qui donne

$M_0 \leq 120 \times 10^6 \times \frac{7{,}840 \times 10^{-6}}{3{,}33 \times 10^{-2}} = 28\,252 \text{ N.m}$

b)

$|2q_s| = \frac{VQ}{I} = \frac{M_0}{2I} \times (0{,}10 \times 0{,}02 \times 0{,}04) = \frac{M_0}{I} \times 4 \times 10^{-5}$

$M_0 \leq \frac{2 \times 38 \times 10^3 \times 7{,}840 \times 10^{-6}}{4 \times 10^{-5}} = 14896 \text{ N.m}$

c)

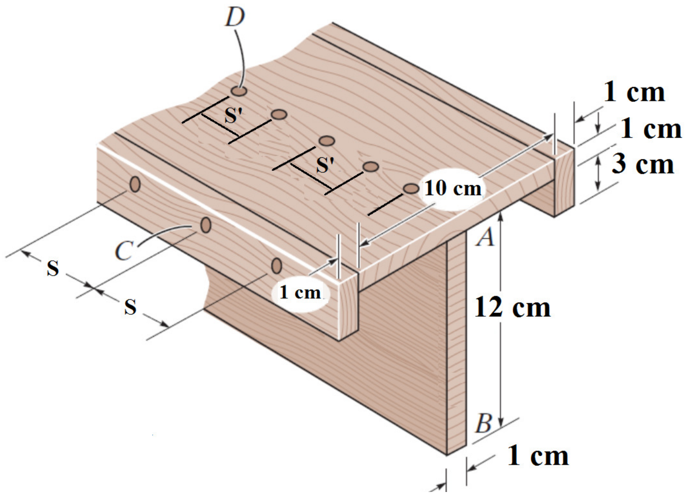
$\rightarrow |q_2| = \frac{VQ}{I} = \frac{M_0}{2I} \times 0{,}01 \times 0{,}040 \times 0{,}050 = \frac{M_0}{I} \times 1{,}0 \times$

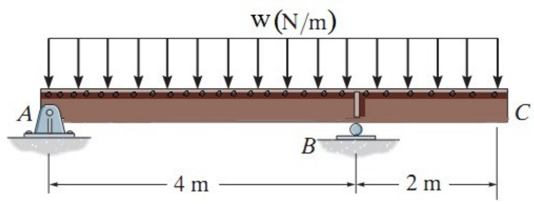
$\frac{M_0}{I} \times 1 \times 10^{-5} \times 0{,}250 < \frac{3}{4}\tau A = \frac{3}{4} 90 \times 10^6 \times \frac{\pi}{4} \times 0{,}006^2$

$M_0 < 5985 \text{ N.m}$

$(M_0)_{max} = 5985 \text{ N.m}$

TBR 4: Maximal force that can be carried by the pins is 450 N (S = 8 cm and S' = 1.5 cm). Find maximal W (1393).





$$\sum M_B = 0 \rightarrow 4A_y - 6w(1) = 0, \quad \rightarrow A_y = 1.5\,w\,(N), \;\; B_y = 4.5w(N)$$

From Statics (draw shear diagram) we have:

$$V_{max} = 2.5w\ N$$ 5

$$\bar{y} = \frac{0.5(10)(1) + 2\times 2(4)(1) + 7(12)(1)}{10\times 1 + 2\times 4\times 1 + 12\times 1} = 3.5\ cm$$ 4

$$I_{NA} = \left[\frac{1}{12}(10)(1)^3 + (10)(1)(3.5-0.5)^2\right] + 2\left[\frac{1}{12}(1)(4)^3 + (4)(1)(3.5-2)^2\right] + \left[\frac{1}{12}(1)(12)^3 + (1)(12)(7-3.5)^2\right] = 410.5\ cm^4$$ 4

$$Q_c = (1)(4)(3.5-2) = 6\ cm^3$$ 5

$$Q_D = (1)(12)(7-3.5) = 42\ cm^3$$ 5

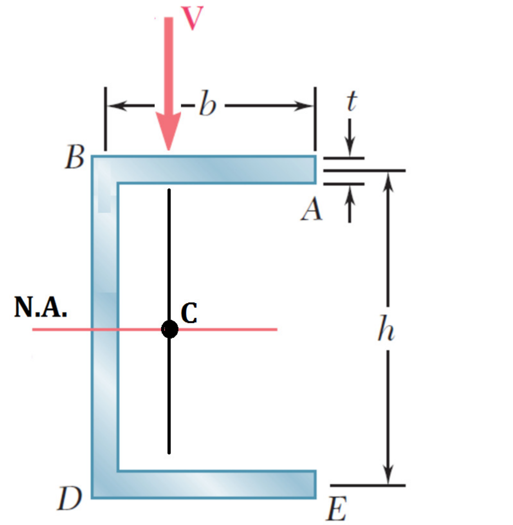
$$F_c = \frac{VQ_c}{I}s \rightarrow 450 = \frac{2.5w(6cm^3)(8cm)}{410.5\ cm^4} \rightarrow w = 1539\ N/m$$ 5

$$F_D = \frac{VQ_D}{I}s' \rightarrow 450 = \frac{2.5w(42cm^3)(1.5cm)}{410.5cm^4} \rightarrow w = 1173\ N/m$$

$$w_{max} = 1173\ N/m$$ 2

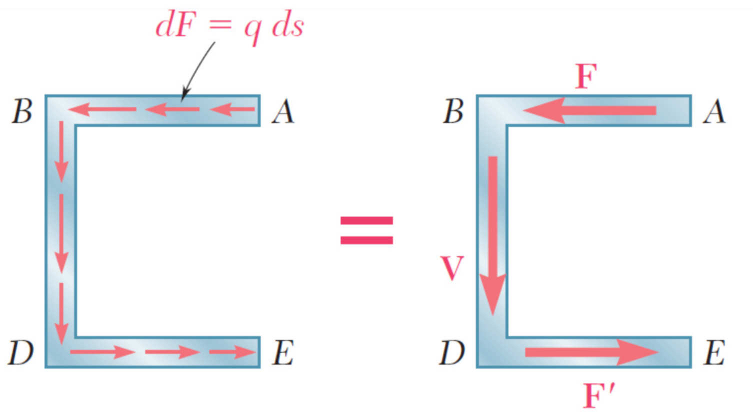
Shear Center

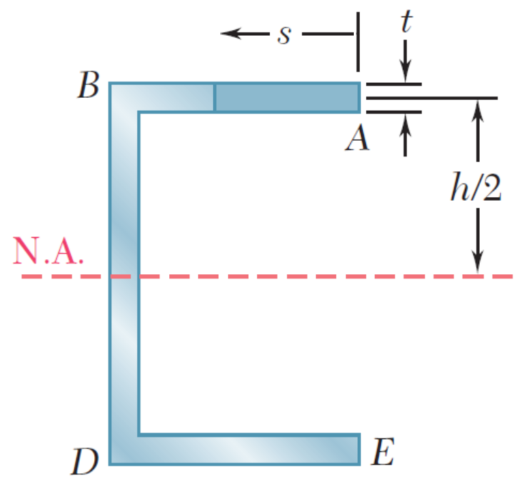
Example 11: Determine the shear flow distribution in the beam with the cross section shown subjected to a shear force of V.



$$q = \frac{VQ}{I}, I = \frac{1}{12}th^3 + 2\left\{\frac{1}{12}bt^3 + bt\left(\frac{h}{2}\right)^2\right\} = \frac{1}{12}th^2(h + 6b)$$

For the flange: $Q = \bar{y}A = \frac{h}{2} \times (ts) \rightarrow q = \frac{V \times \frac{h}{2} ts}{I} = \frac{Vhts}{2I}$



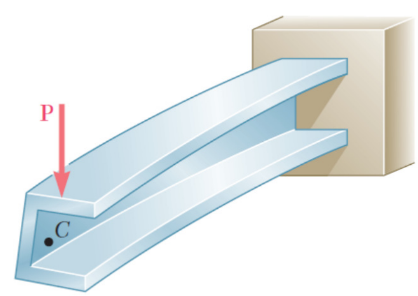


$$F = \int_0^b q \times ds = \int_0^b \frac{Vhts}{2I} \times ds = \frac{Vht}{2I}\frac{s^2}{2}\bigg|_0^b = \frac{Vthb^2}{4I}$$

$$F = F' = \frac{Vthb^2}{4I} \quad \left(\sum F_x = 0\right)$$

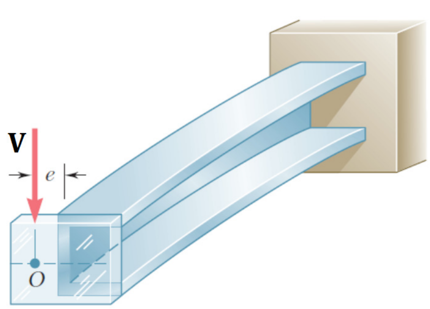
$$V = \int_B^D q_{web} \times ds' \; \left(\sum F_y = 0\right),\; BUT \;\; \sum T \neq 0$$

→ FBD is wrong → There should be an internal twisting moment. Therefore → $q \neq VQ/I \rightarrow q = VQ/I + q_{twisting}$

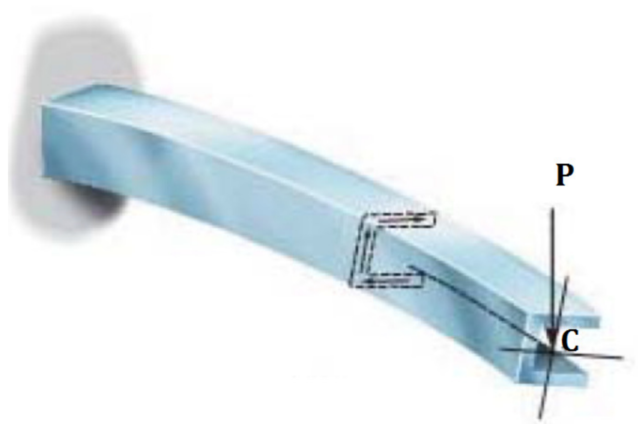


To eliminate twisting when bending : $Ve = Fh$

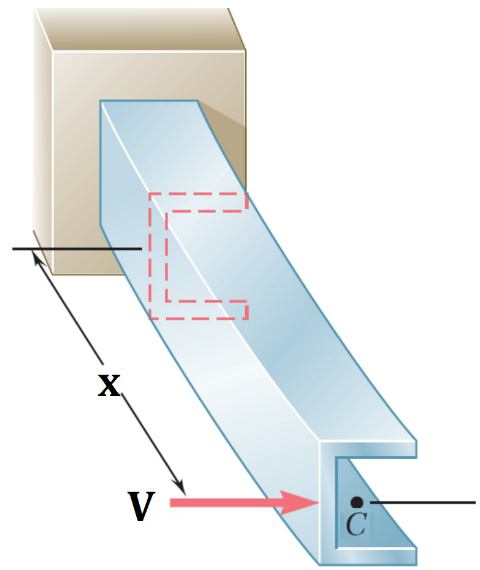
$$\rightarrow e = \frac{Fh}{V} \rightarrow e = \frac{\frac{Vthb^2}{4I}h}{V} = \frac{3b^2}{(h + 6b)}$$



Placement of load to eliminate twisting.



Important Points Regarding the Shear Center Concept

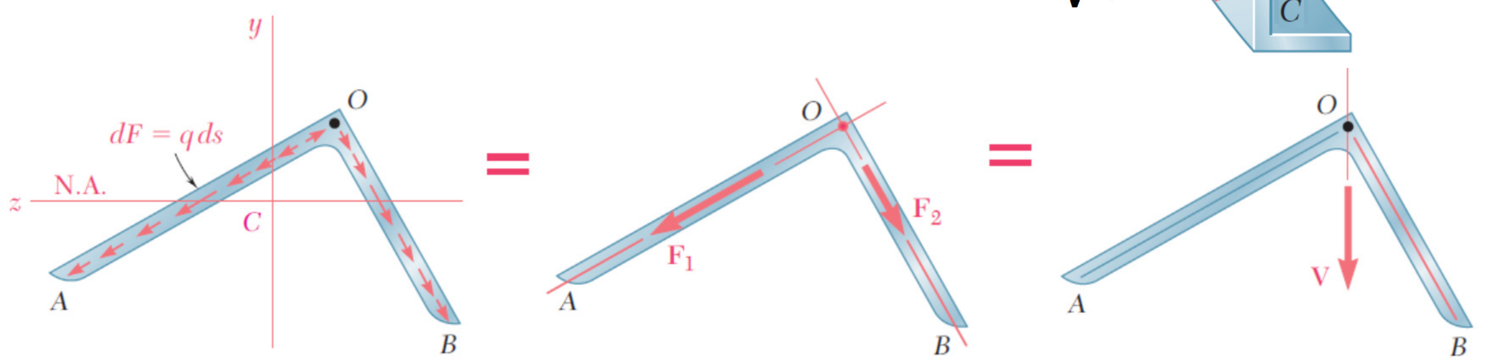


1- Point O is called the shear center.

2- The location of shear center is only a function of the geometry of the cross section.

3- The shear center always lies on an axis of symmetry of the cross section (see the figure).

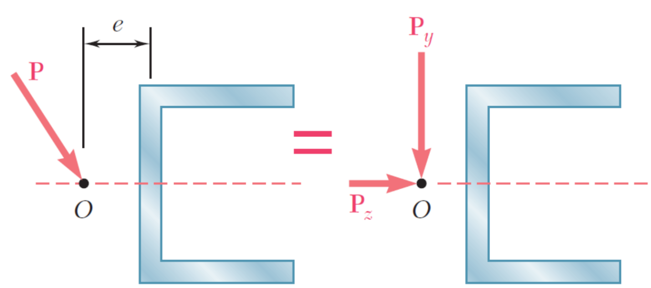
4- For the beam shown below the shear center is located at point O.

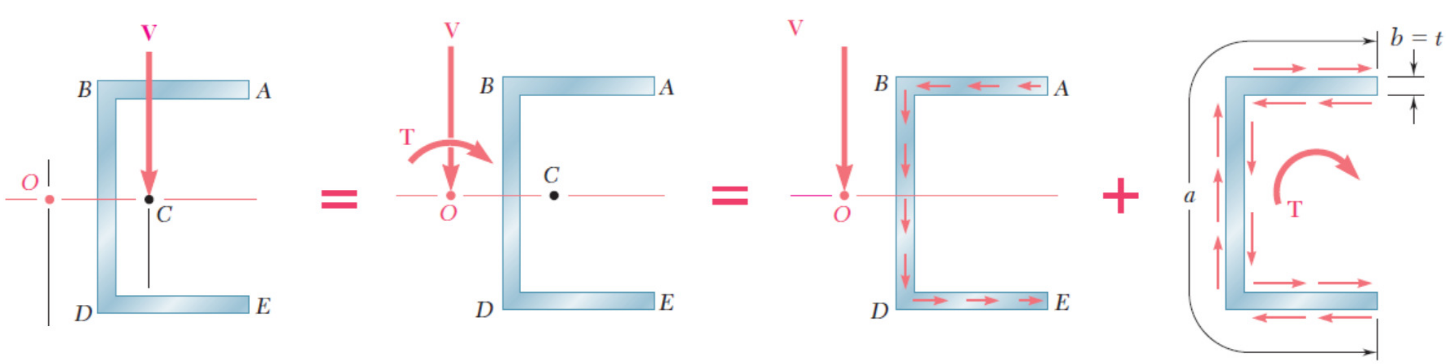


(*a*) Shear stresses (*b*) Resultant forces on elements (*c*) Placement of **V** to eliminate twisting

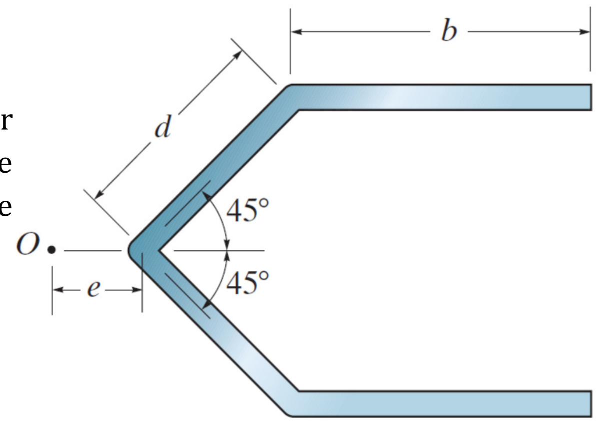
5- In the case of an oblique load P, the member will also be free of any twist if the load P is applied at the shear center of the section (see the figure).

6- Combined stresses can be calculated using shear formula developed in this chapter and torsion formula developed in chapter 3 (see below and refer to Example 6.07 of Beer and Johnston, 6th version).





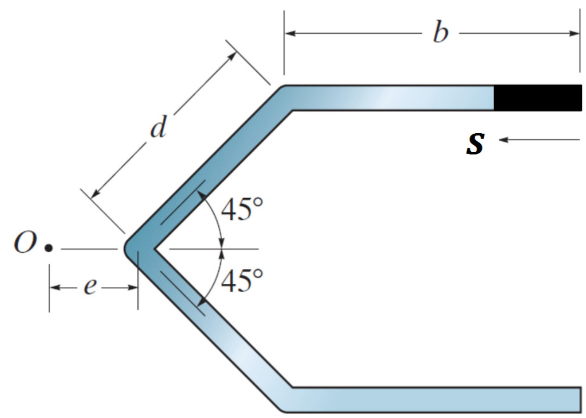
Example 12: Determine the location e of the shear center, point O, for the thin-walled member having the cross section shown. The member segments have the same thickness t.



$$I=\frac{1}{12}\left(\frac{t}{\sin 45°}\right)(2d\sin 45°)^3+2(bt(d\sin 45°)^2)$$

$$=\frac{td^2}{3}(d+3b)$$

$$Q=\bar{y}A=(d\sin 45°)(st)=(td\sin 45°)\,s$$



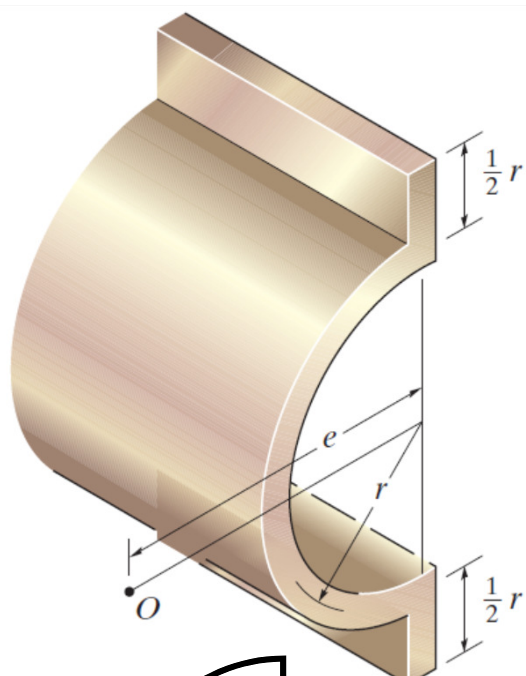
$$q=\frac{VQ}{I}=\frac{V(td\sin 45°)\,s}{\frac{td^2}{3}(d+3b)}=\frac{3V\sin 45°}{d(d+3b)}s$$

$$F=\int_0^b q\,ds=\int_0^b\frac{3V\sin 45°}{d(d+3b)}s\,ds=\frac{3b^2\sin 45°}{2d(d+3b)}V$$

$$V\times e=F(2d\sin 45°)=\frac{3b^2\sin 45°}{2d(d+3b)}V(2d\sin 45°)$$

$$e=\frac{3b^2}{2(d+3b)}$$

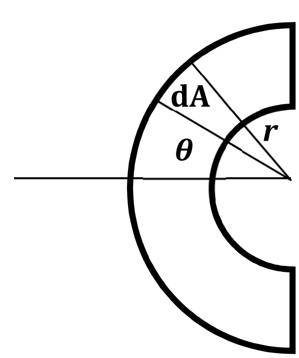
TBR 5: Determine the location e of the shear center, point O, for the thin-walled member having the cross section shown. The member segments have the same thickness t.



$$I=2\left\{\frac{1}{12}t\left(\frac{1}{2}r\right)^3+\frac{1}{2}rt\left(r+\frac{1}{4}r\right)^2\right\}+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(r\sin\theta)^2(rt\,d\theta)\rightarrow$$

$$I=3.15413\,tr^3$$

$$Q=\frac{1}{2}rt\left(r+\frac{1}{4}r\right)+\int_{\theta}^{\frac{\pi}{2}}(r\sin\theta)(rt\,d\theta)=(0.625+\cos\theta)tr^2$$



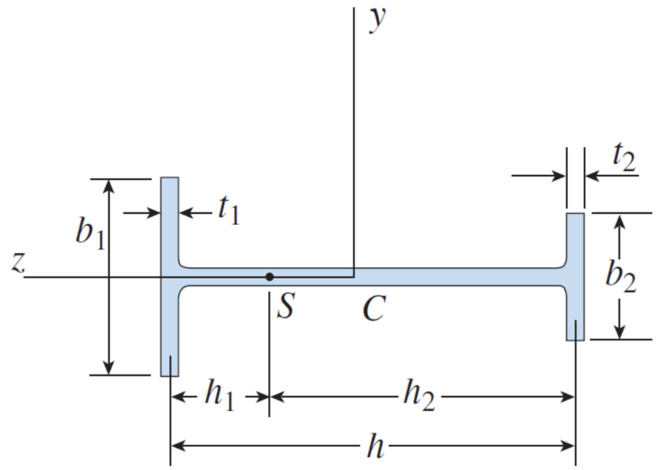
$$q=\frac{VQ}{I}=\frac{V(0.625+\cos\theta)tr^2}{3.15413\,tr^3}=\frac{V(0.625+\cos\theta)}{3.15413\,r}$$

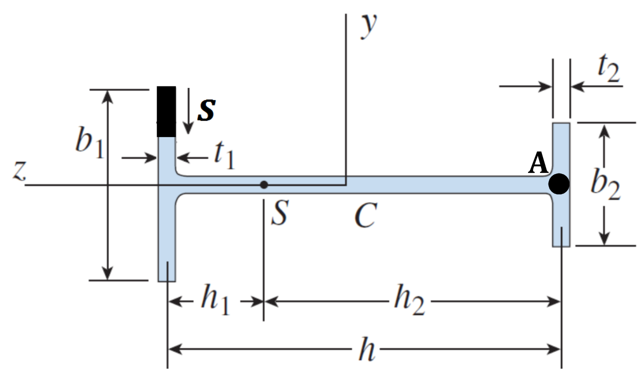
$$V\times e=r\times\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(q\,r d\theta)=\frac{Vr}{3.15413}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(0.625+\cos\theta)\,d\theta\rightarrow$$

$$e=1.26\,r$$

Example 13: The cross section of an unbalanced wide-flange beam is shown in the figure. Derive the following formula for the distance h_1 from the centerline of one flange to the shear center S:

$$h_1 = \frac{t_2 b_2^3 h}{t_1 b_1^3 + t_2 b_2^3}$$





$$for\ the\ left\ flange: q_1 = \frac{VQ_1}{I} = \frac{V(\frac{b_1}{2} - \frac{s}{2}) \times st_1}{I}$$

$$F_1 = \int_0^{b_1} q\, ds = \int_0^{b_1} \frac{V(b_1 - s) \times st_1}{2I}\, ds = \frac{Vb_1^3 t_1}{12I}$$

$$\sum M_A = 0 \rightarrow F_1 h = Vh_2 \rightarrow h_2 = \frac{\frac{Vb_1^3 t_1}{12I} h}{V} = \frac{b_1^3 t_1}{12I} h$$

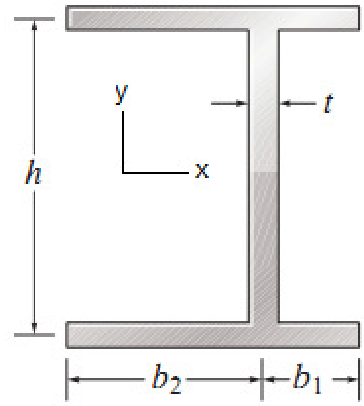
$$h_1 = h - h_2 \ \ and \ \ I = \frac{1}{12} t_1 b_1^3 + \frac{1}{12} t_2 b_2^3$$

$$\rightarrow h_1 = \frac{t_2 b_2^3 h}{b_1^3 t_1 + b_2^3 t_2}$$

$$For\ b_1 = b_2 \ \ and \ \ t_1 = t_2 \rightarrow h_1 = \frac{h}{2}$$

Note that shear flow in the web (horizontal part) is zero because if you cut it at any section Q becomes equal to zero.

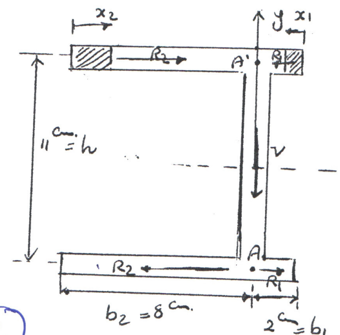
TBR 6: A beam with the cross section shown undergoes a downward shear force of 34.5 kN. Find center of shear. If this shear force is applied at the center of shear, find distribution of shear flow on the cross section as well as magnitude and location of maximal shear stress (1391).



$b_1 = 2\ cm, \qquad b_2 = 8\ cm,$

$t = 1\ cm\ \ and\ \ h = 11\ cm$

$I_x = \frac{1}{12}(10)(12)^3 - \frac{1}{12}(9)(10)^3 = 690\ cm^4$ (2.)



$R_1 = \int_A dR_1 = \int \tau_1\, dA = \int_0^{b_1} \tau_1\, t\, dx_1$

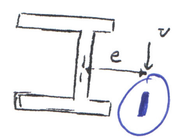
$\boxed{R_1 = \int_0^{b_1} q_1\, dx_1}$; $\boxed{R_2 = \int_0^{b_2} q_2\, dx_2}$

$q_1 = \frac{VQ_1}{I}$

$Q_1 = (t x_1)\frac{h}{2} = \frac{th}{2} x_1 \;\Big\} \rightarrow q_1 = \frac{Vth}{2I} x_1$ (2)

$q_2 = \frac{VQ_2}{I}$

$Q_2 = (t x_2)\frac{h}{2} = \frac{th}{2} x_2 \;\Big\} \rightarrow q_2 = \frac{Vth}{2I} x_2$ (2)



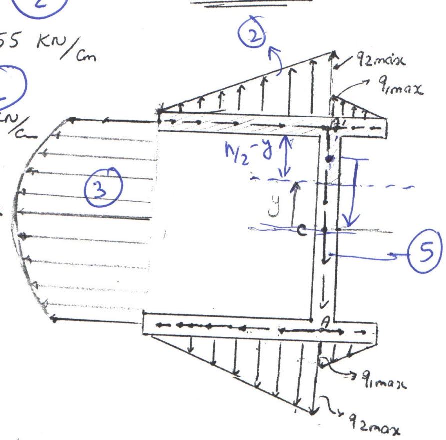
$R_1 = \frac{Vth}{2I} \cdot \frac{b_1^2}{2}$ (1) ; $R_2 = \frac{Vth}{2I} \cdot \frac{b_2^2}{2}$ (1)

$\Sigma \overset{+}{\curvearrowright} M_A = 0 \longrightarrow R_2 \cdot h - R_1 h = V \cdot e$ (2) $\longrightarrow e = \frac{th^2}{4I}(b_2^2 - b_1^2) = 2.63\ cm$ (1)

$q_1 = \frac{Vth}{2I} x_1$; $q_{1max} = \frac{Vthb_1}{2I} = 0.55\ KN/cm$ (2)

$q_2 = \frac{Vth}{2I} x_2$; $q_{2max} = \frac{Vthb_2}{2I} = 2.2\ KN/cm$ (2)

$q_A = q_{A'} = q_1 + q_2 = 2.75\ KN/cm$ (2)



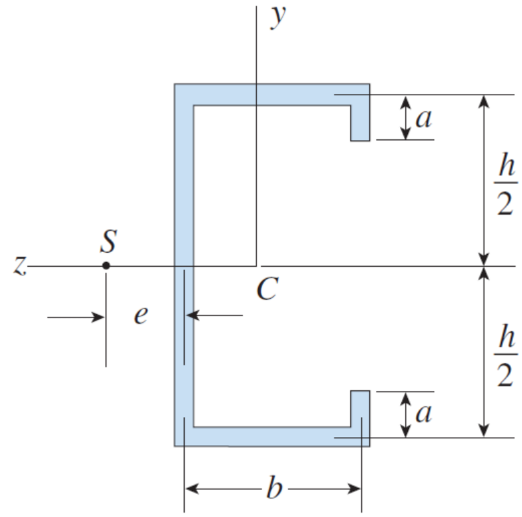
$q_3 = \frac{VQ_3}{I}$; $Q_3 = (b_1+b_2)t.\frac{h}{2} + (\frac{h}{2} - y)t \times \frac{1}{2}(\frac{h}{2} + y)$

$q_3 = \frac{Vt}{2I}\left[(b_1+b_2)h + (\frac{h^2}{4} - y^2)\right]$ (5)

$q_{max} = q_{3max} = q_C = q_3(y=0) = 3.5063\ KN/cm$

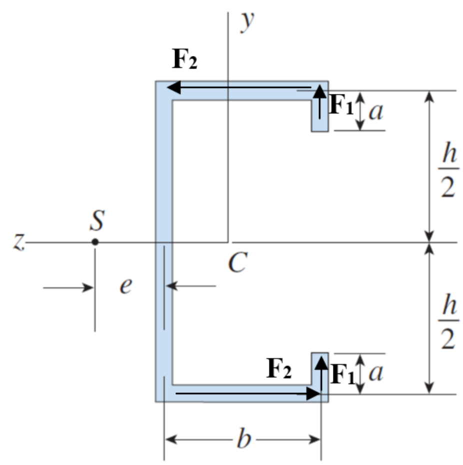
$\tau_{max} = \tau_C = q_C/t = 3.5063\ KN/cm^2 = 35.063\ Mpa$ (2)

TBR 7: Find center of shear for thin wall beam shown as functions of a, b, and h. If the shear force of V is applied to the center of shear, determine position and magnitude of the maximal shear stress (1392).



$$I = 2\frac{1}{12}th^3 + 2bt\left(\frac{h}{2}\right)^2 - \frac{1}{12}t(h-2a)^3 = \frac{th^3}{6} + \frac{bth^2}{2} - \frac{t(h-2a)^3}{12}$$

For the small vertical parts: $Q = st\left(\frac{h}{2} - a + \frac{s}{2}\right)$,



$$q = \frac{VQ}{I} = \frac{stV\left(\frac{h}{2} - a + \frac{s}{2}\right)}{I}$$

$$F_1 = \int_0^a \frac{stV\left(\frac{h}{2} - a + \frac{s}{2}\right)}{I} ds = \frac{tV}{I}\int_0^a s\left(\frac{h}{2} - a + \frac{s}{2}\right) ds$$

$$= \frac{tV}{I}\left(\frac{h}{4}a^2 - \frac{a^3}{2} + \frac{a^3}{6}\right) = \frac{tVa^2}{12I}(3h-4a)$$

For horizontal parts: $Q = st\left(\frac{h}{2}\right) + at\left(\frac{h}{2} - \frac{a}{2}\right)$

$$q = \frac{VQ}{I} = \frac{V\left(st\left(\frac{h}{2}\right) + at\left(\frac{h}{2} - \frac{a}{2}\right)\right)}{I}$$

$$F_2 = \int_0^a \frac{V\left(st\left(\frac{h}{2}\right) + at\left(\frac{h}{2} - \frac{a}{2}\right)\right)}{I} ds = \frac{tV}{I}\int_0^b \left(s\left(\frac{h}{2}\right) + a\left(\frac{h}{2} - \frac{a}{2}\right)\right) ds = \frac{tV}{I}\left(\frac{h}{4}b^2 + \frac{ah - a^2}{2}b\right)$$

$$= \frac{btV}{4I}(hb + 2ah - 2a^2)$$

$$\boldsymbol{Ve = 2F_1 b + F_2 h} \rightarrow Ve = 2\frac{tVa^2}{12I}(3h-4a)b + \frac{btV}{4I}(hb + 2ah - 2a^2)h$$

$$e = \frac{bt\left(6ah^2 + 3h^2b - 8a^3\right)}{12I}$$

$$I = \frac{th^3}{6} + \frac{bth^2}{2} - \frac{t(h-2a)^3}{12} = \frac{t}{12}(2h^3 + 6bh^2 - (h-2a)^3) \rightarrow e = \frac{b(6ah^2 + 3h^2b - 8a^3)}{2h^3 + 6bh^2 - (h-2a)^3}$$

$$\boldsymbol{\tau_{max}(at\ N.A.)} = \boldsymbol{\frac{VQ_C}{It}} = \frac{V\left(bt\left(\frac{h}{2}\right) + at\left(\frac{h}{2} - \frac{a}{2}\right) + t\frac{h}{2}\frac{h}{4}\right)}{It} = \frac{V\left(12bh + 12a(h-a) + 3h^2\right)}{2t\left(2h^3 + 6bh^2 - (h-2a)^3\right)}$$