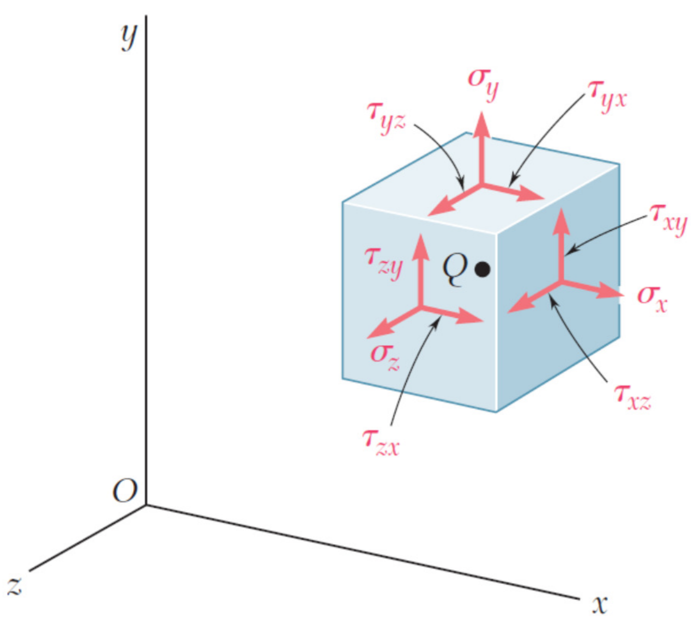
CHAPTER 7: Transformation of Stress/Strain and Thin-walled Pressure Vessels

Chapter 7 is concerned with finding normal and shear stresses acting on inclined sections cut through a member, because these stresses may be larger than those on a stress element aligned with the cross section. Normal and shear stresses in beams, shafts, and bars can be calculated from the basic formulas discussed in the preceding chapters. For instance, the stresses in a beam are given by the flexure and shear formulas ($\sigma = My/I$ and $\tau = VQ/It$), and the stresses in a shaft are given by the torsion formula ($\tau = Tr/J$). The stresses calculated from these formulas act on cross sections of the members, but larger stresses may occur on **inclined sections (*As we saw in Chapter 1 for uniaxial loading*)**. Therefore, we will begin our analysis of stresses and strains by discussing methods for finding the normal and shear stresses acting on inclined sections cut through a member.

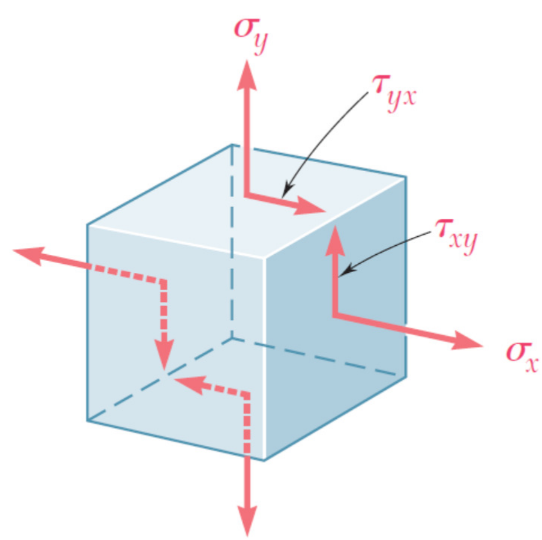
We have already derived expressions for the normal and shear stresses acting on inclined sections in both uniaxial stress and pure shear. In the case of uniaxial stress, we found that the maximum shear stresses occur on planes inclined at 45° to the axis, whereas the maximum normal stresses occur on the cross sections. In the case of pure shear, we found that the maximum tensile and compressive stresses occur on 45° planes. In an analogous manner, the stresses on inclined sections cut through a beam may be larger than the stresses acting on a cross section. To calculate such stresses, we need to determine the stresses acting on inclined planes under a more general stress state known as **plane stress**. The objectives of Chapter 7 are therefore twofold: 1- finding the state of stress at a given point of a structure under general loading and 2- finding maximal normal and shear stresses at that point.

Plane Stress

Our discussion of the transformation of stress will deal mainly with plane stress, i.e., with a situation in which two of the faces of the cubic element are free of any stress



General state of stress

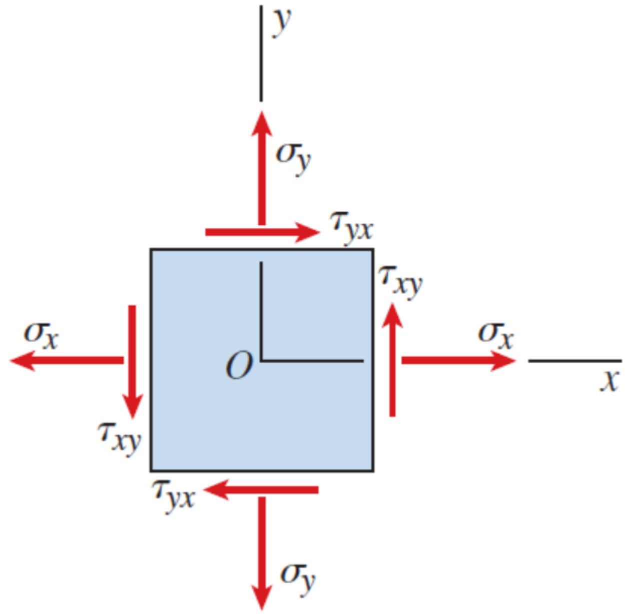


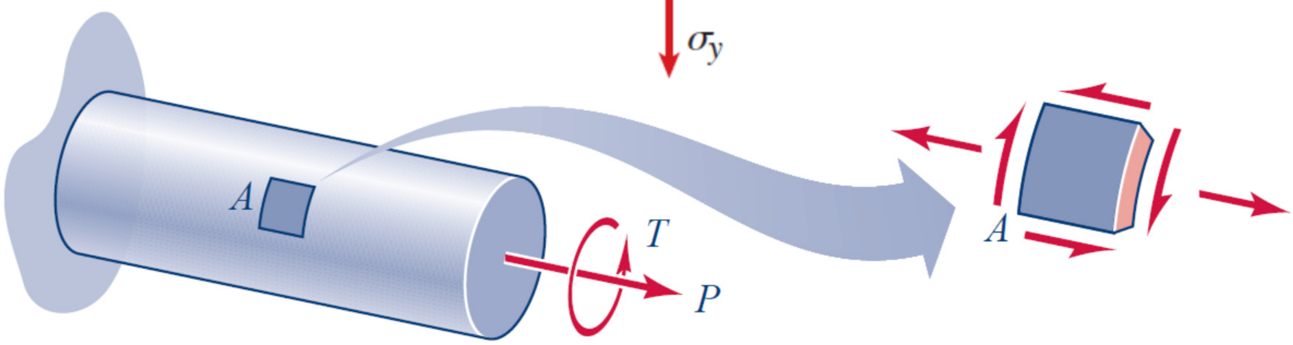
Plane stress

$$\sigma_z = \tau_{xz} = \tau_{zx} = \tau_{yz} = \tau_{zy} = 0$$

$$\varepsilon_z \neq 0 \; (\varepsilon_z = \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E})$$

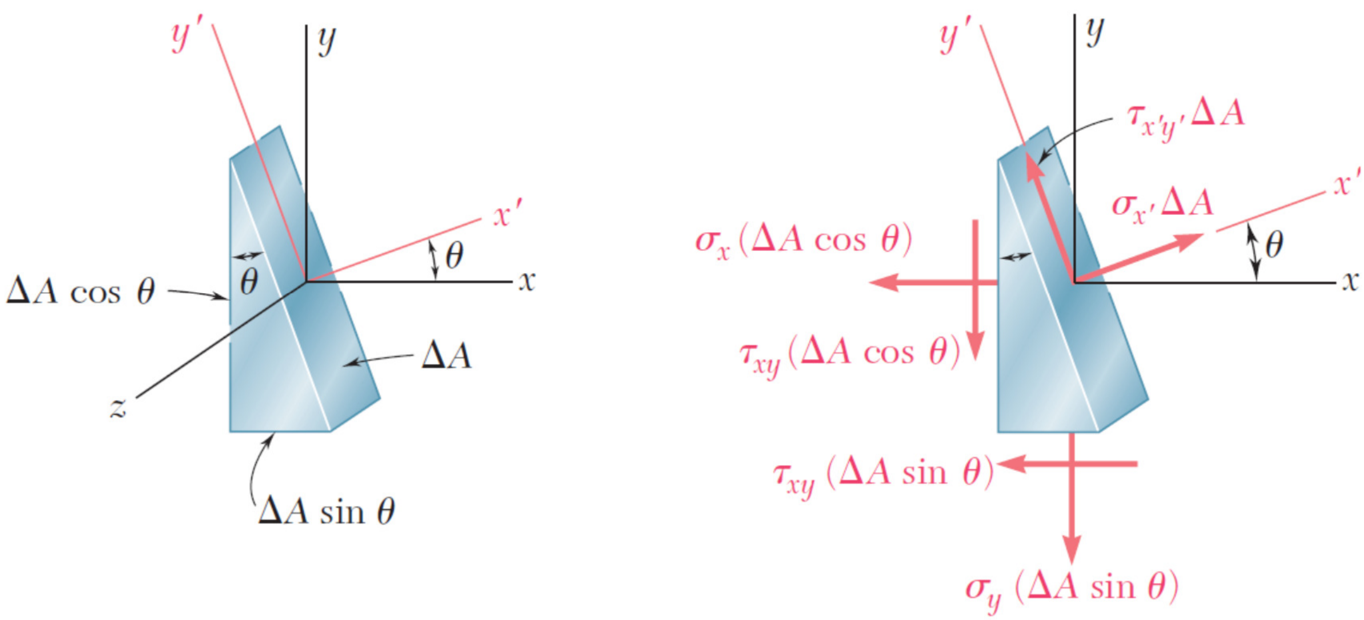
Plane stress



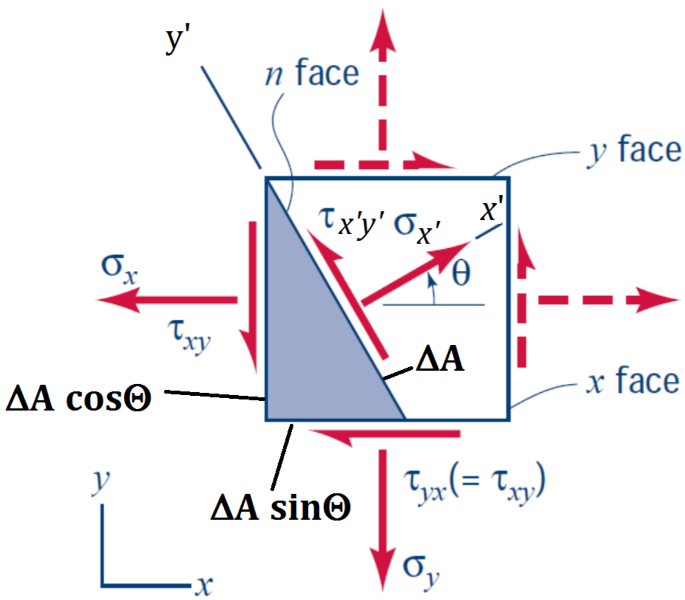


An example of plane stress at point A of a rod under axial load and torque

TRANSFORMATION OF PLANE STRESS



Using components along the *x*' and *y*' axes, we write the following equilibrium equations:



$$\sum F_{x'} = 0 \rightarrow \sigma_{x'}\Delta A - \sigma_x(\Delta A\, cos\theta)cos\theta - \tau_{xy}(\Delta A\, cos\theta)sin\theta - \sigma_y(\Delta A\, sin\theta)sin\theta - \tau_{xy}(\Delta A\, sin\theta)cos\theta = 0$$

$$\sum F_{y'} = 0 \rightarrow \tau_{x'y'}\Delta A + \sigma_x(\Delta A\, cos\theta)sin\theta - \tau_{xy}(\Delta A\, cos\theta)cos\theta - \sigma_y(\Delta A\, sin\theta)cos\theta + \tau_{xy}(\Delta A\, sin\theta)sin\theta = 0$$

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\, sin\theta\, cos\theta$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y)sin\theta\, cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

Recalling the trigonometric relations:

$$sin\, 2\theta = 2\, sin\theta\, cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\boldsymbol{\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\, sin\, 2\theta} \quad (1)$$

$$\boldsymbol{\tau_{x'y'} = -\frac{(\sigma_x - \sigma_y)}{2} sin2\theta + \tau_{xy}\, \cos 2\theta} \quad (2)$$

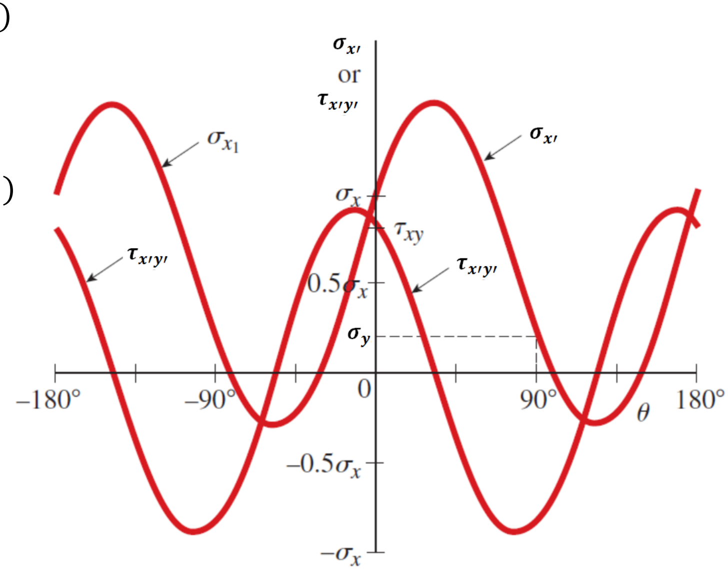
Stress transformation Equations for plane stress

Replacing Θ with Θ+90^o:

$$\boldsymbol{\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\, sin\, 2\theta} \quad (3)$$

We also note that: $\boldsymbol{\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}}$

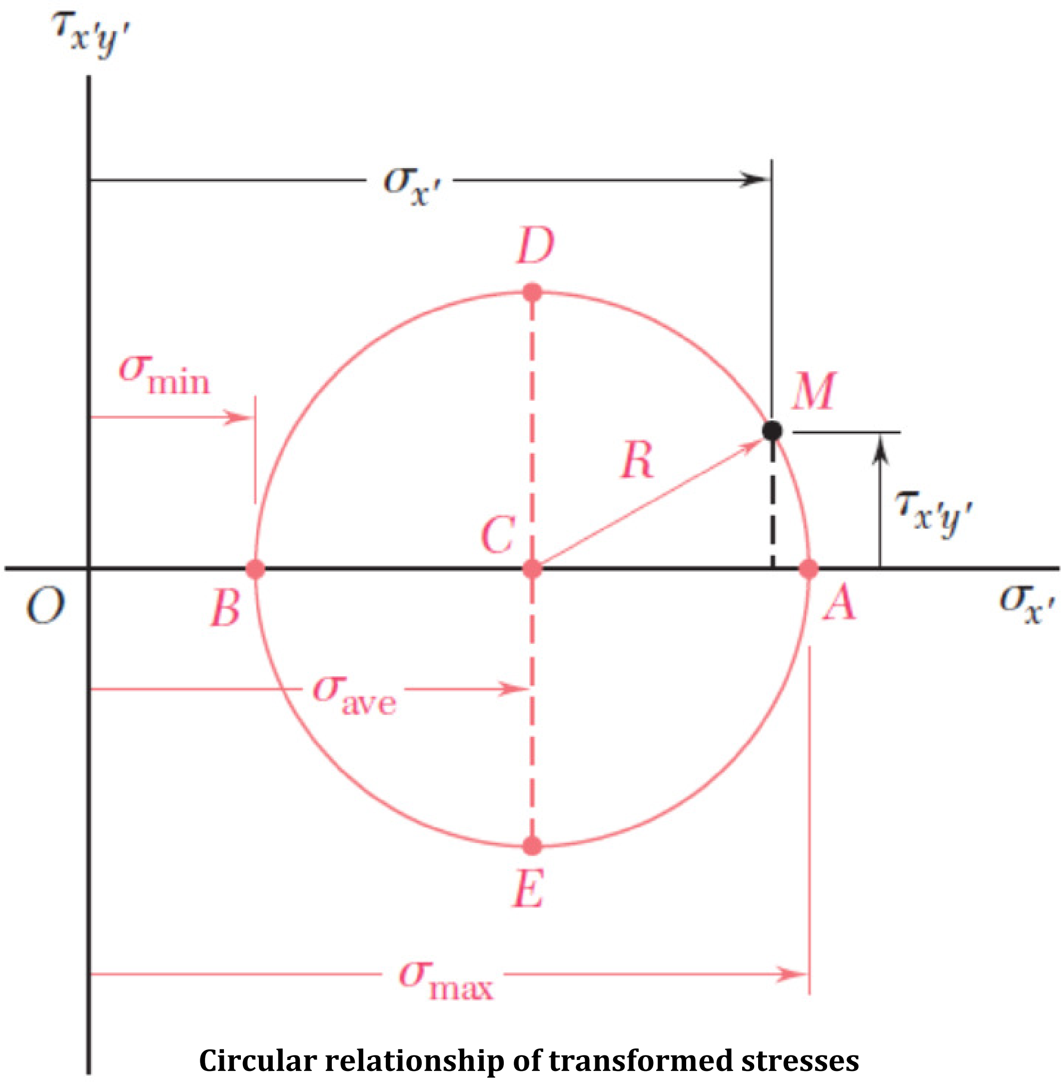
This equation shows that the sum of the normal stresses acting on perpendicular faces of plane-stress elements (at a given point in a stressed body) is constant and independent of the angle Θ. Remember that the angle Θ in these equations is measured counterclockwise from the x face to the x' face.



Principal Stresses; Maximum Shearing Stress

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}, \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \rightarrow \left(\sigma_{x'} - \sigma_{avg}\right)^2 + \tau_{x'y'}^2 = R^2$$



Circular relationship of transformed stresses

The two points *A* and *B* where the circle intersects the horizontal axis are of special interest: Point *A* corresponds to the maximum value of the normal stress $\sigma_{x'}$, while point *B* corresponds to its minimum value. Besides, both points correspond to a zero value of the shearing stress $\tau_{x'y'}$. Thus, the values Θ_p of the parameter Θ which correspond to points *A* and *B* can be obtained by setting $\tau_{x'y'} = 0$:

$$\tau_{x'y'} = -\frac{(\sigma_x - \sigma_y)}{2} sin2\theta + \tau_{xy} \cos 2\theta = 0 \rightarrow \tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

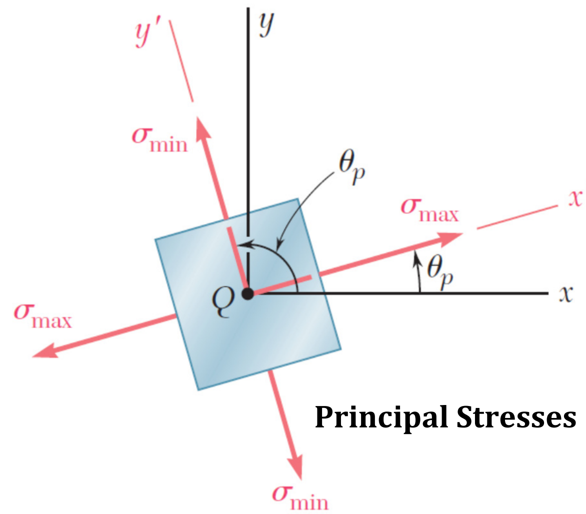
This equation defines two values $2\Theta_P$ that are 180^O apart, and thus two values Θ_P that are 90^O apart. Either of these values can be used to determine the orientation of the corresponding element below:

Alternatively we could determine ΘP as follow:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\frac{d\sigma_{x'}}{d\theta} = 0 \rightarrow -(\sigma_x - \sigma_y) \sin 2\theta + 2\tau_{xy} \cos 2\theta = 0$$

$$\tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$



Principal Stresses

The planes containing the faces of the element obtained in this way are called the *principal planes of stress* at point Q. We observe that principal stress can be calculated as follow:

$$\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_{min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Unless it is possible to tell by inspection which of the two principal planes is subjected to σ_{max} and which is subjected to σ_{min}, it is necessary to substitute one of the values ΘP into the **Stress transformation Equations** in order to determine which of the two corresponds to the maximum value of the normal stress.
Referring again to the circle, we note that the points D and E located on the vertical diameter of the circle correspond to the largest numerical value of the shearing stress $\tau_{x'y'}$. Since the abscissa of points D and E is $\sigma_{avg} = (\sigma_x + \sigma_y)/2$ the values ΘS of the parameter Θ corresponding to these points are obtained by setting $\sigma_{x'} = (\sigma_x + \sigma_y)/2$:

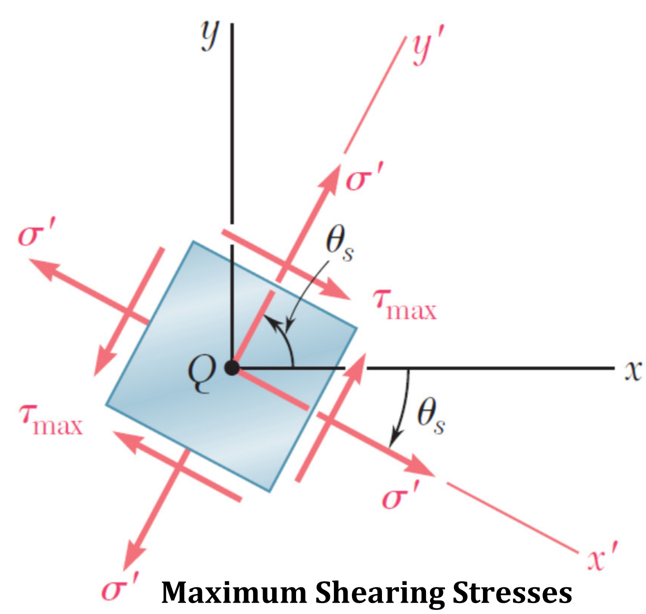
$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta = \frac{\sigma_x + \sigma_y}{2} \rightarrow \tan 2\theta_S = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

Alternatively we could determine ΘS by setting $\frac{d\tau_{x'y'}}{d\theta} = 0$. This equation defines two values 2ΘS that are 180^o apart, and thus two values ΘS that are 90^o apart. Either of these values can be used to determine the orientation of the element corresponding to the maximum shearing stress as below:

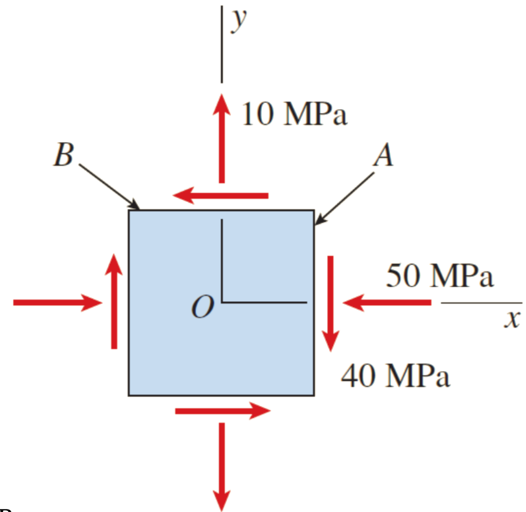
The maximum value of the shearing stress is equal to the radius *R* of the circle:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_{max} - \sigma_{min}}{2}$$

The normal stress corresponding to the condition of maximum shearing stress is:
$\sigma' = \sigma_{avg} = (\sigma_x + \sigma_y)/2$
Note that: $\tan 2\theta_P \times \tan 2\theta_s = -1$ means that: $\theta_P = \theta_S \pm 45°$. We thus conclude that the planes of maximum shearing stress are at 45^o to the principal planes. This confirms the results obtained earlier in the case of a centric axial loading and in the case of torsional loading.



Maximum Shearing Stresses



Example 1: At a point on the surface of a generator shaft the stresses are as shown below. Determine the following quantities: (a) the stresses acting on an element inclined at an angle ϴ = 45°, (b) the principal stresses, and (c) the maximum shear stresses.

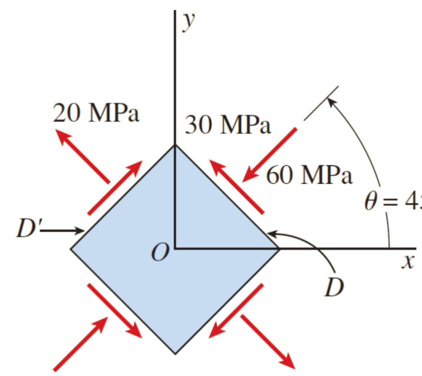
$\sigma_x = -50\ MPa, \sigma_y = +10\ MPa, \tau_{xy} = -40\ MPa$

$$\sigma_{x\prime} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\, sin\, 2\theta$$

$$\sigma_{x\prime} = \frac{-50+10}{2} + \frac{-50-10}{2}\cos 2(45°) + (-40) sin\, 2(45°) = -60\ MPa$$

$$\sigma_{y\prime} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\, sin\, 2\theta = \frac{-50+10}{2} - \frac{-50-10}{2}\cos 2(45°) - (-40) sin\, 2(45°) = 20\ MPa$$

$$\tau_{x\prime y\prime} = -\frac{(\sigma_x - \sigma_y)}{2} sin 2\theta + \tau_{xy}\ \cos 2\theta = -\frac{-50-10}{2} sin\ 2(45°) + (-40)\ \cos 2(45°) = 30\ MPa$$



$$\sigma_{max,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

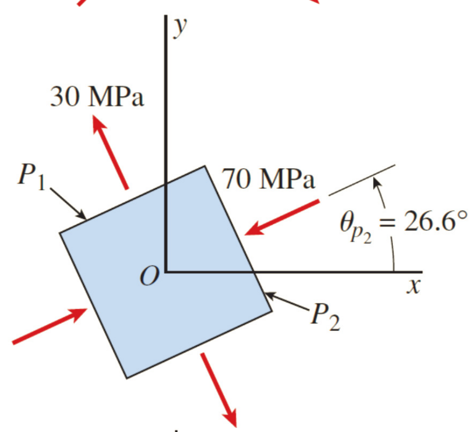
$$= \frac{-50+10}{2} \pm \sqrt{\left(\frac{-50-10}{2}\right)^2 + (-40)^2} = +30\ MPa, -70\ MPa$$

$$\tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-40)}{-(50)-(10)} = \frac{4}{3} \rightarrow \theta_P = 26.56°$$

$$\textbf{\textit{Check}}: \sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \frac{-50+10}{2} + \frac{-50-10}{2}\cos 2(26.56°) + (-40)\sin 2(26.56°)$$

$$= -70\ MPa$$



$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{-50+10}{2} - \frac{-50-10}{2}\cos 2(26.56°) - (-40)\sin 2(26.56°)$$

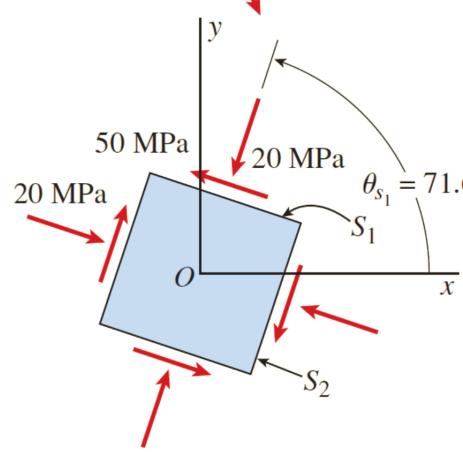
$$= 30\ MPa$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{-50-10}{2}\right)^2 + (-40)^2} = \pm 50\ MPa$$

$$\tan 2\theta_S = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{-(-50-10)}{2(-40)} = -\frac{3}{4} \rightarrow \theta_S = -18.4°$$

$$\textbf{\textit{Check}}: \tau_{x'y'} = -\frac{(\sigma_x - \sigma_y)}{2} sin 2\theta + \tau_{xy}\ \cos 2\theta$$

$$= -\frac{-50-10}{2} sin\ 2(-18.4°) + (-40)\ \cos 2(-18.4°) = -50\ MPa$$

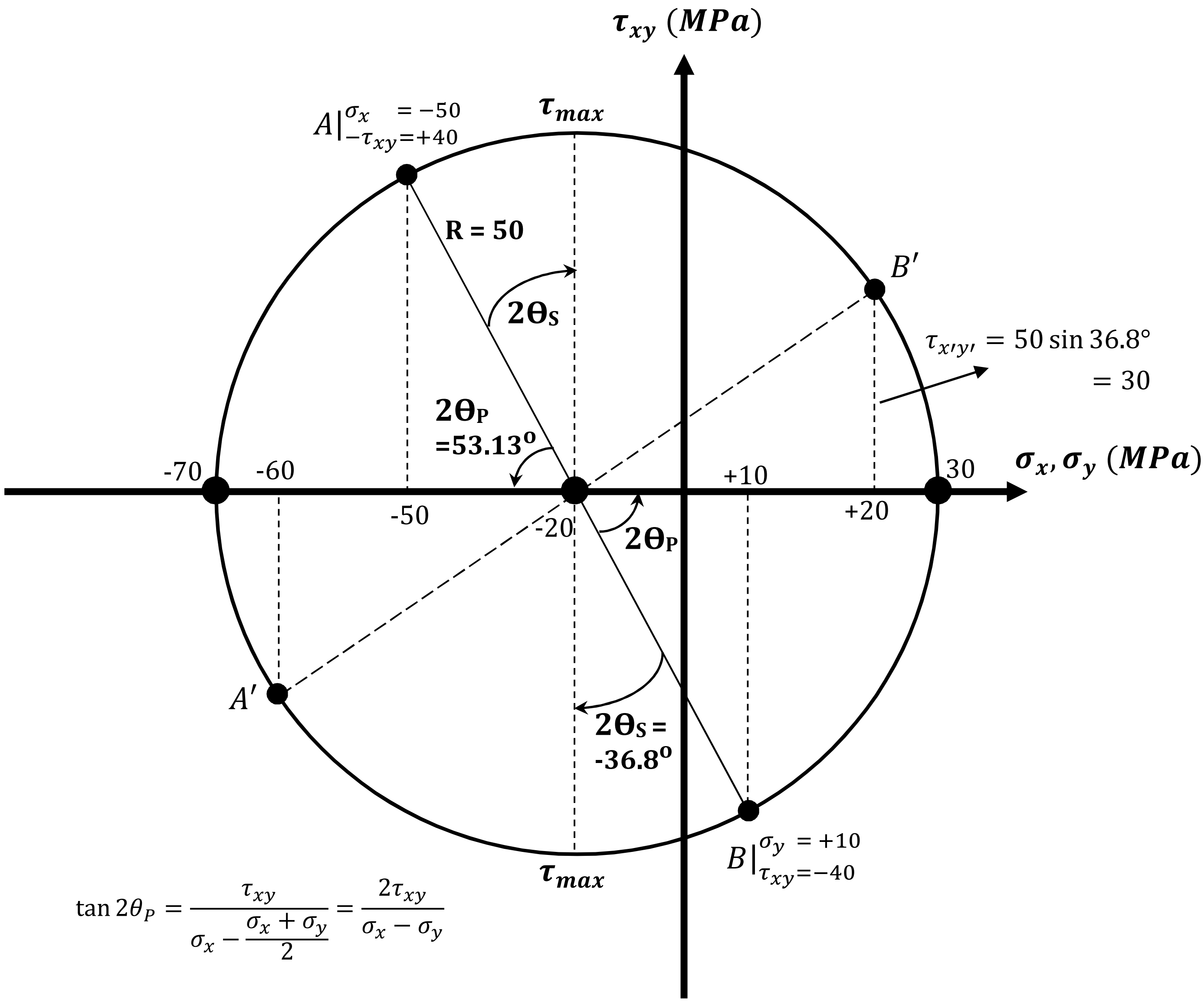


Example 2: Use Mohr's circle to solve Example 1.

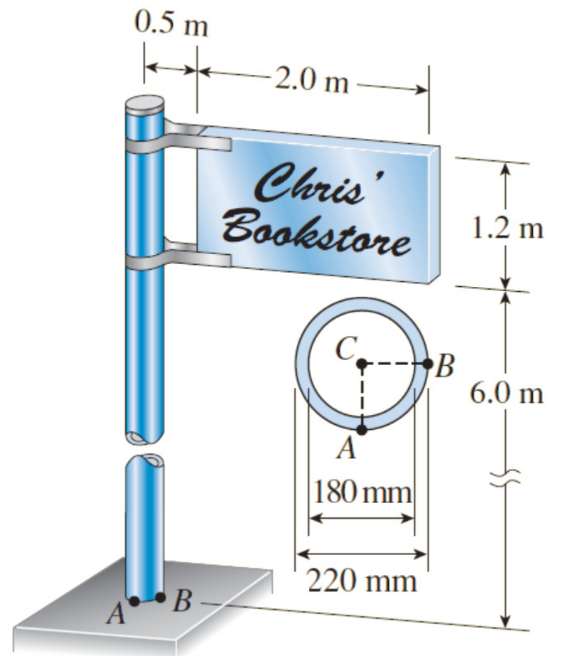
(1) Find center of Mohr's circle $(\sigma_{avg} = \frac{\sigma_x+\sigma_y}{2}, 0)$, $for\ this\ example$ $(-20{,}0)$

(2) Find points $(\sigma_x\ ,\ -\tau_{xy})$ and $(\sigma_y\ ,\ +\tau_{xy})$. ***These two pints are located on the circle.***

(3) Plot the circle using these three points

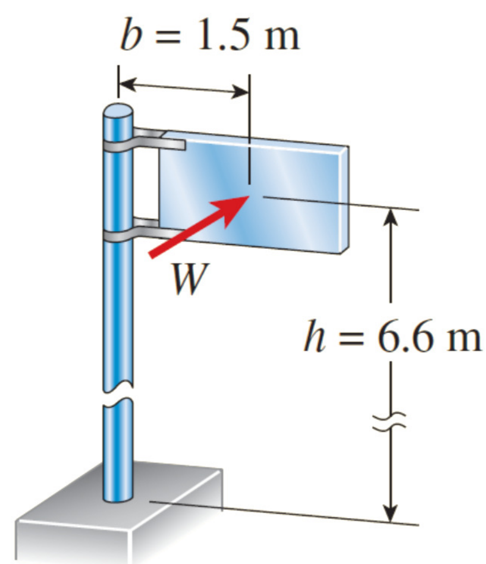


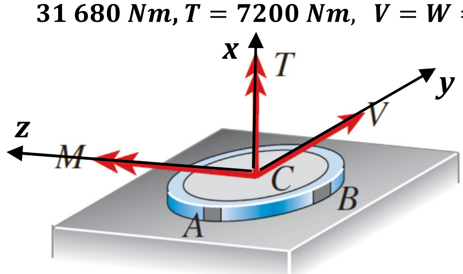
$$\tan 2\theta_P = \frac{\tau_{xy}}{\sigma_x - \frac{\sigma_x+\sigma_y}{2}} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

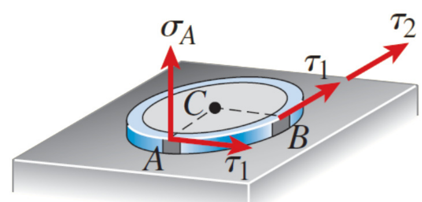


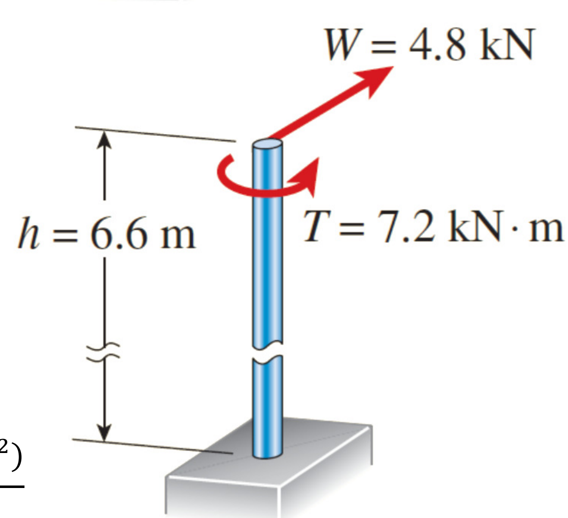
Example 3: A sign of dimensions 2.0 m × 1.2 m is supported by a hollow circular pole having outer diameter 220 mm and inner diameter 180 mm as shown. The sign is offset 0.5 m from the centerline of the pole and its lower edge is 6.0 m above the ground. Determine the principal stresses and maximum shear stresses at points *A* and *B* at the base of the pole due to a wind pressure of 2.0 kPa against the sign.

The wind pressure against the sign produces a resultant force *W* that acts at the midpoint of the sign and is equal to the pressure *P* times the area *A* over which it acts: $\boldsymbol{W = PA = 2000\ Pa \times (2 \times 1.2\ m^2) = 4800\ N}$. The line of action of this force is at height *h* = 6.6 m above the ground and at distance *b* = 1.5 m from the centerline of the pole. The wind force acting on the sign is statically equivalent to a lateral force *W* and a torque *T* acting on the pole. The torque is equal to the force *W* times the distance *b*: $\boldsymbol{T = Wb = 4800\ N \times (1.5\ m) = 7200\ Nm}$. The stress resultants at the base of the pole consist of a bending moment *M*, a torque *T*, and a shear force *V*. Their magnitudes are: $\boldsymbol{M = Wh = 4800\ N \times 6.6\ m = 31\,680\ Nm, T = 7200\ Nm,\ \ V = W = \ 4800\ N}$.









$$\sigma_A = \frac{Mc_A}{I} = \frac{(31\,680\,000\ Nmm)(\frac{220}{2}\ mm)}{\frac{\pi}{4}(110^4 - 90^4)} = 54.91\ MPa$$

$$\tau_A(torsion) = \frac{Tr_A}{J} = \frac{(7200\,000\ Nmm)\left(\frac{220}{2}\ mm\right)}{\frac{\pi}{2}(110^4 - 90^4)} = 6.24\ MPa$$

$$\tau_A(shear) = 0$$

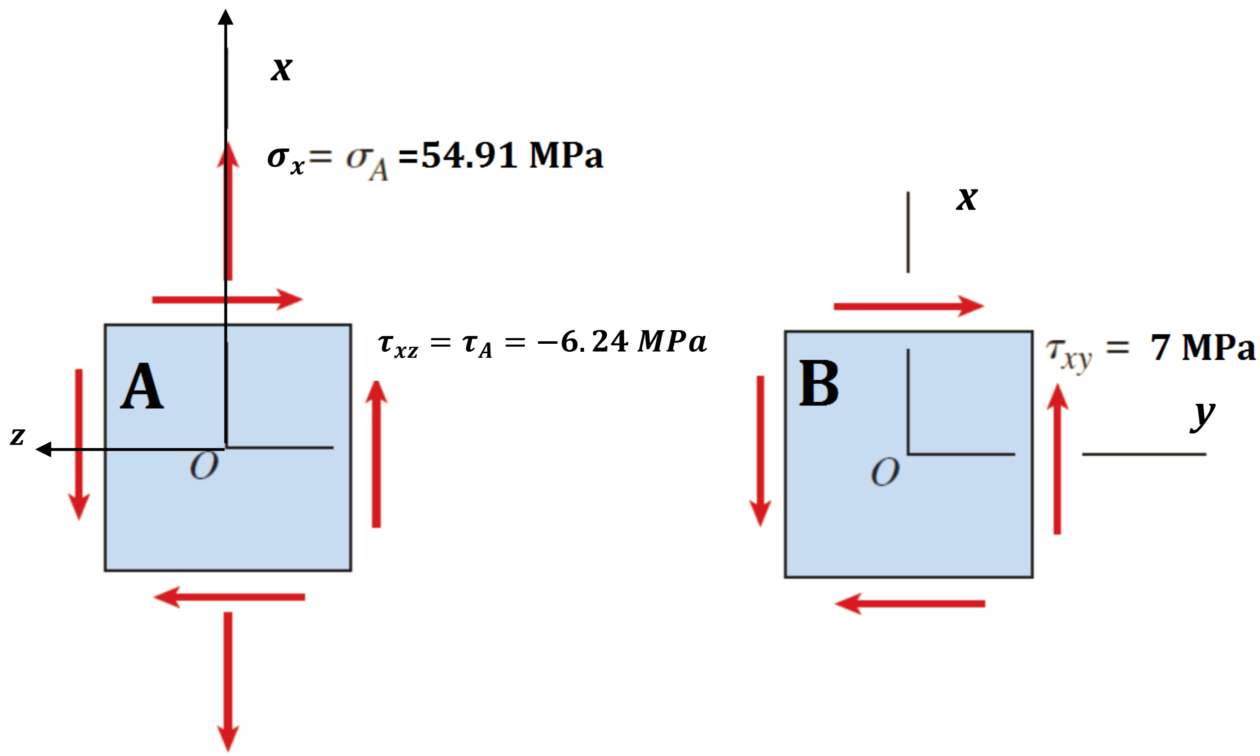
$$\sigma_B = 0, \tau_B(torsion) = \frac{Tr_B}{J} = \frac{(7200\,000\ Nmm)(\frac{220}{2}\ mm)}{\frac{\pi}{2}(110^4 - 90^4)}$$

$$= 6.24\ MPa$$

$$\tau_B(shear) = \frac{VQ}{It} = \frac{(4800\ N)(\frac{4 \times 110}{3\pi} \times \frac{\pi}{2}110^2 - \frac{4 \times 90}{3\pi} \times \frac{\pi}{2}90^2)}{\frac{\pi}{4}(110^4 - 90^4) \times (220 - 180)}$$

$$= 0.76\ MPa$$

$$\tau_B = 6.24\ MPa + 0.76\ MPa = 7\ MPa$$



Principal stresses and maximum shearing stress at A:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} = \frac{54.91 + 0}{2} \pm \sqrt{\left(\frac{54.91 - 0}{2}\right)^2 + (-6.24)^2} = 55.7\ MPa, -0.7\ MPa$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{54.91 - 0}{2}\right)^2 + (-6.24)^2} = 28.2\ MPa$$

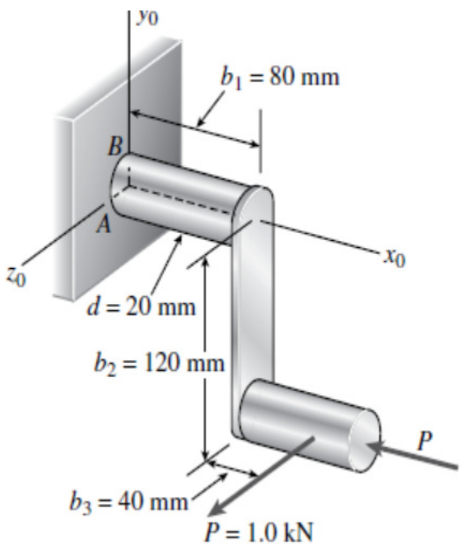
Principal stresses and maximum shearing stress at B:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{0 + 0}{2} \pm \sqrt{\left(\frac{0 - 0}{2}\right)^2 + 7} = 7\ MPa, -7\ MPa$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{0 - 0}{2}\right)^2 + (7)^2} = 7\ MPa$$

Keep in mind that only the effects of the wind pressure are considered in this analysis. Other loads such as the weight of the structure that also produce stresses at the base of the pole have been neglected.

TBR 1: Determine maximal tensile, compressive, and shear stresses at points A and B



$A = \frac{\pi}{4} d^2 \qquad A = 314.159 \text{ mm}^2$

$I = \frac{\pi}{64} d^4 \qquad I = 7.854 \times 10^3 \text{ mm}^4$

$I_p = 2I \qquad I_p = 1.571 \times 10^4 \text{ mm}^4$

$Q = \frac{2}{3} r^3 \; Q = 666.667 \text{ mm}^3$

STRESS RESULTANTS AT THE SUPPORT

$V_x = P$ (Axial force in X-dir.)

$V_y = 0$ (Shear force in Y-dir.)

$V_z = P$ (Shear force in Z-dir.)

$M_x = Pb_2$ (Torsional Moment)

$M_x = 120 \text{ kN} \cdot \text{mm}$

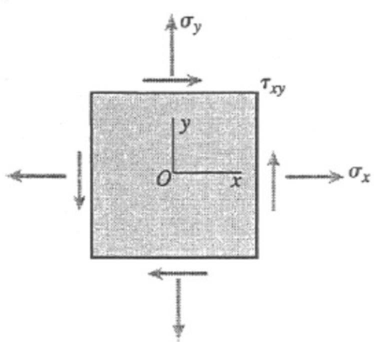
$M_y = P(b_1 + b_3)$ (Bending Moment)

$M_y = 120 \text{ kN} \cdot \text{mm}$

$M_z = Pb_2$(Bending Moment)

$M_z = 120 \text{ kN} \cdot \text{mm}$

(a) STRESSES AT POINT *A*



$$\sigma_x = -\frac{V_x}{A} - \frac{M_y}{I} r$$

$\sigma_x = -155.972$ MPa (compressive)

$\sigma_y = 0$

$\tau_{xy} = \frac{M_x d}{2 I_p} \qquad \tau_{xy} = 76.394$ MPa

$$\sigma_1 = \frac{\sigma_x - \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

MAX. TENSILE STRESS $\sigma_1 = 31.2$ MPa ←

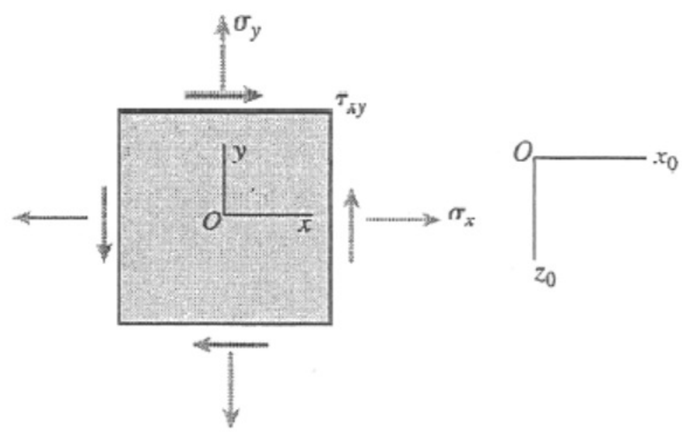
MAX. COMPRESSIVE STRESS

$\sigma_2 = -187.2$ MPa ←

MAX. SHEAR STRESS $\tau_{max} = 109.2$ MPa ←

(b) STRESSES AT POINT *B*

$$\sigma_x = -\frac{V_x}{A} + \frac{M_z}{I} r$$



$\sigma_x = 149.606$ MPa (tensile)

$\sigma_y = 0$

$\tau_{xy} = \frac{M_x d}{2 I_p} - \frac{V_z Q}{I d}$ **= 76.39 - 4.24 = 72.15 MPa**

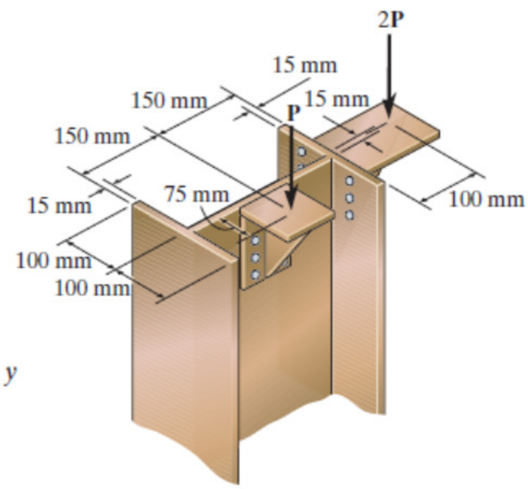
MAX. TENSILE STRESS $\sigma_1 =$ **178.7 MPa** ←

MAX. COMPRESSIVE STRESS

$\sigma_2 =$ **-29.1 MPa** ←

MAX. SHEAR STRESS $\tau_{max} =$ **103.9 MPa**

TBR 2: If P = 60 kN, determine the maximum normal stress developed on the cross section of the column.



Equivalent Force System: Referring to Fig. *a*,

$+\uparrow \Sigma F_x = (F_R)_x; \quad -60 - 120 = -F \quad F = 180\text{ kN}$

$\Sigma M_y = (M_R)_y; \quad -60(0.075) = -M_y \quad M_y = 4.5\text{kN}\cdot\text{m}$

$\Sigma M_z = (M_R)_z; \quad -120(0.25) = -M_z \quad M_z = 30\text{kN}\cdot\text{m}$

Section Properties: The cross-sectional area and the moment of inertia about the y and z axes of the cross section are

$$A = 0.2(0.3) - 0.185(0.27) = 0.01005\text{ m}^2$$

$$I_z = \frac{1}{12}(0.2)(0.3^3) - \frac{1}{12}(0.185)(0.27^3) = 0.14655(10^{-3})\text{ m}^4$$

$$I_y = 2\left[\frac{1}{12}(0.015)(0.2^3)\right] + \frac{1}{12}(0.27)(0.015^3) = 20.0759(10^{-6})\text{m}^4$$

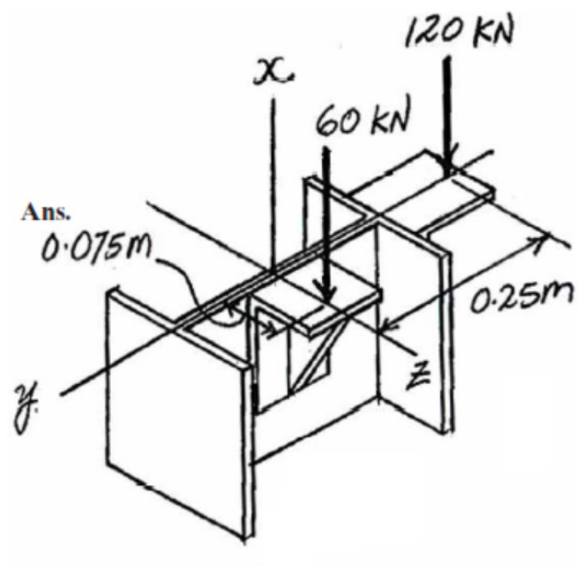
Normal Stress: The normal stress is the combination of axial and bending stress. Here, **F** is negative since it is a compressive force. Also, $\mathbf{M}_y$ and $\mathbf{M}_z$ are negative since they are directed towards the negative sense of their respective axes. By inspection, point A is subjected to a maximum normal stress. Thus,

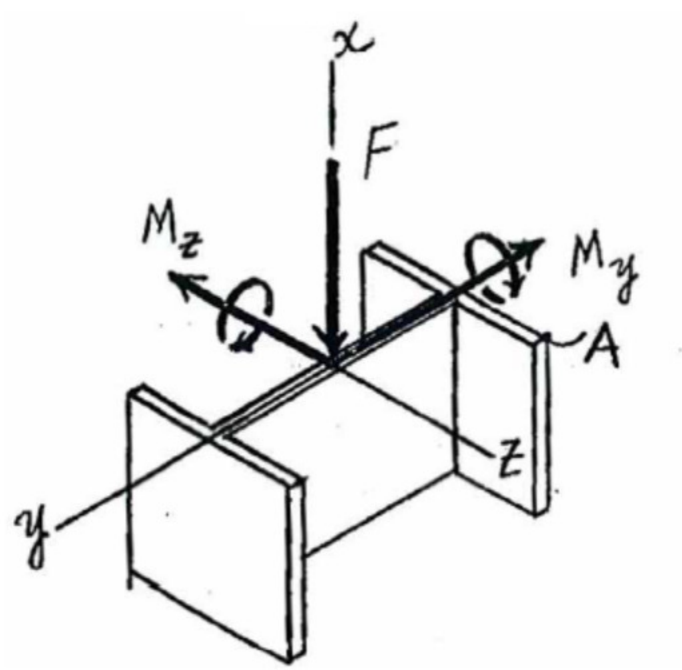
$$\sigma = \frac{N}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$\sigma_{\max} = \sigma_A = \frac{-180(10^3)}{0.01005} - \frac{[-30(10^3)](-0.15)}{0.14655(10^{-3})} + \frac{[-4.5(10^3)](0.1)}{20.0759(10^{-6})}$$

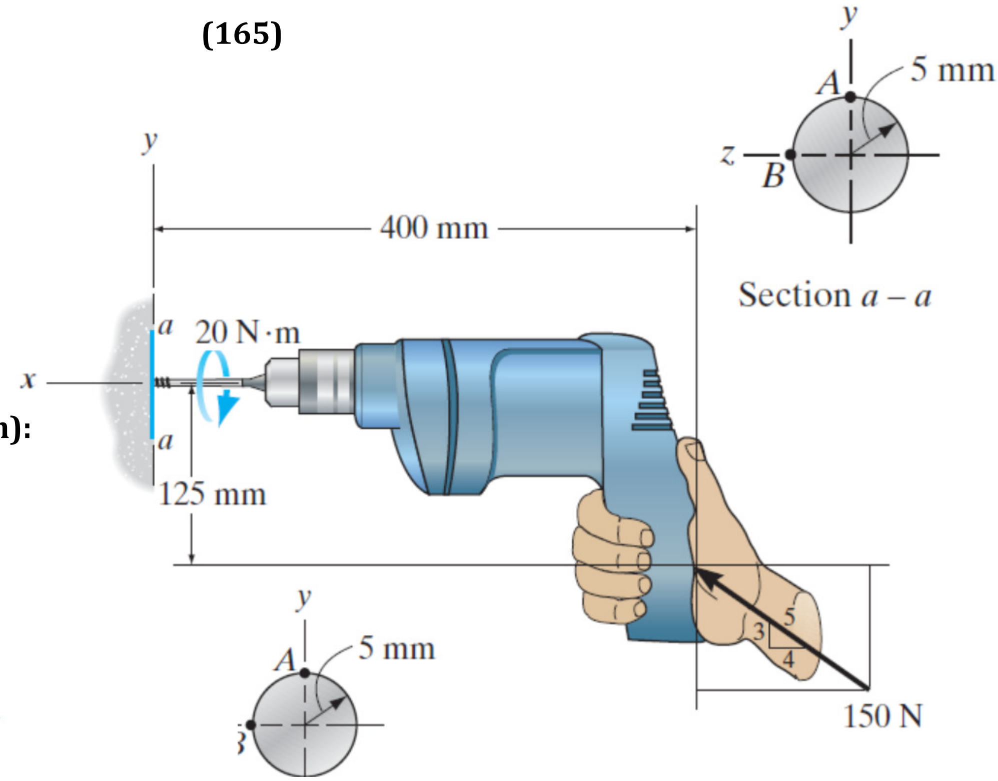
$$= -71.0\text{ MPa} = 71.0\text{ MPa(C)}$$

Ans.



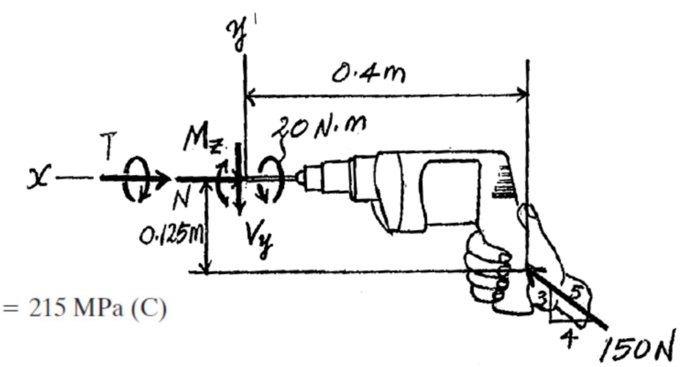


TBR 3: The drill is jammed in the wall and is subjected to the torque and force shown. Determine the principal stresses at points A and B on the cross section of drill bit at section a–a (1390).



From Statics (see free body diagram):

$$N = 120\,\text{N} \quad V_y = 90\,\text{N}$$
$$T = 20\,\text{N}\cdot\text{m} \quad M_z = 21\,\text{N}\cdot\text{m}$$



$$A = \pi\left(0.005^2\right) = 25\pi\left(10^{-6}\right)\text{m}^2$$
$$I_z = \frac{\pi}{4}\left(0.005^4\right) = 0.15625\pi\left(10^{-9}\right)\text{m}^4$$
$$J = \frac{\pi}{2}\left(0.005^4\right) = 0.3125\pi\left(10^{-9}\right)\text{m}^4$$
$$Q_A = 0$$
$$Q_B = \bar{y}'A' = \frac{4(0.005)}{3\pi}\left[\frac{\pi}{2}\left(0.005^2\right)\right] = 83.333\left(10^{-9}\right)\text{m}^3$$

POINT A:

$$\sigma = \frac{N}{A} - \frac{M_z y}{I_z} = \frac{-120}{25\pi\left(10^{-6}\right)} - \frac{21(0.005)}{0.15625\pi\left(10^{-9}\right)} = -215.43\text{ MPa} = 215\text{ MPa (C)}$$

$$\left[\left(\tau_{xz}\right)_V\right]_A = \frac{V_y Q_A}{I_z t} = 0 \qquad \left[\left(\tau_{xz}\right)_T\right]_A = \frac{Tc}{J} = \frac{20(0.005)}{0.3125\pi\left(10^{-9}\right)} = 102\text{ MPa}$$

POINT B:

$$\sigma = \frac{N}{A} - \frac{M_z y}{I_z} = \frac{-120}{25\pi\left(10^{-6}\right)} - 0 = -1.528\text{ MPa} = 1.53\text{ MPa(C)}$$

$$\left[\left(\tau_{xy}\right)_V\right]_B = \frac{V_y Q_B}{I_z t} = \frac{90\left[83.333\left(10^{-9}\right)\right]}{0.15625\pi\left(10^{-9}\right)(0.01)} = 1.528\text{ MPa}$$

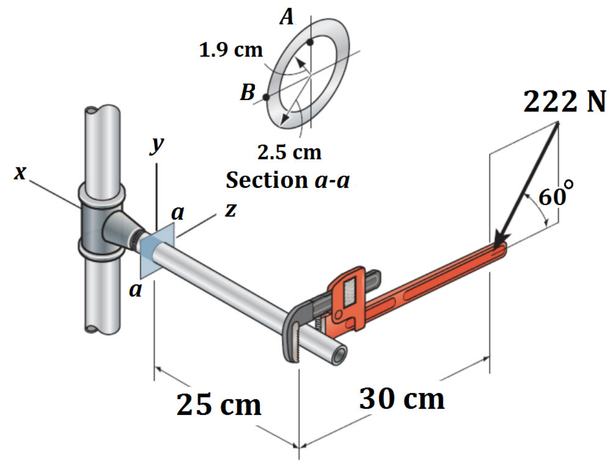
$$\left[\left(\tau_{xy}\right)_T\right]_B = \frac{T_C}{J} = \frac{20(0.005)}{0.3125\pi\left(10^{-9}\right)} = 101.86\text{ MPa}$$

$$\left(\tau_{xy}\right)_B = \left[\left(\tau_{xy}\right)_T\right]_B - \left[\left(\tau_{xy}\right)_V\right]_B = 101.86 - 1.528 = 100.33\text{ MPa} = 100\text{ MPa}$$

From formula or Mohr's circle:

$$\sigma_{min,max})_A = -255.91 \text{ and } 40.49\ MPa$$
$$\sigma_{min,max})_B = -101.1 \text{ and } 99.5\ MPa$$
$$\left|\tau_{\text{ max})A}\right| = 148.2\ MPa$$
$$\left|\tau_{\text{ max})B}\right| = 100.3\ MPa$$

TBR 4: Find principal stresses and maximal shear stress at points A and B of the *a-a* section (1391).



$$\sum F_y = 0 \rightarrow V_y = 222\ \sin 60° = 192.25\ N$$

$$\sum F_z = 0 \rightarrow V_z = 222\ \cos 60° = 111\ N$$

$$\sum M_x = 0 \rightarrow T + (222\ \sin 60°) \times 300\ mm = -57677.3\ Nmm$$

$$\sum M_y = 0 \rightarrow M_y - (222\ \cos 60°) \times 250\ mm = 27750\ Nmm$$

$$\sum M_z = 0 \rightarrow M_z + (222\ \sin 60°) \times 250\ mm = -48064.4\ Nmm$$

$$I_y = I_z = \frac{\pi}{4}(25^4 - 19^4) = 204442.3\ mm^4$$

$$J = 2I = 408884.6\ mm^4$$

Stress at point A:

$$\sigma_x = \frac{M_z\,(19\ mm)}{I_z} = \frac{(48064.4\ Nmm)(19\ mm)}{204442.3\ mm^4} = 4.46\ MPa$$

$$\tau_{xz} = \frac{Tr_A}{J} = \frac{(57677.3\ Nmm)(19\ mm)}{408884.6\ mm^4} = 2.68\ MPa$$

$$\tau_{xy})_{Vy} = 0$$

$$\tau_{xz})_{Vz} = \frac{V_z Q_A}{I_y t} = \frac{(111\ N)(\frac{4\times 25}{3\pi}\frac{\pi}{2}25^2 - \frac{4\times 19}{3\pi}\frac{\pi}{2}19^2)}{(204442.3\ mm^4)((25-19)\times 2)} = 0.26\ MPa$$

$$\tau_{xz})A = 2.68 - 0.26 = 2.42\ MPa$$

Stress at point B:

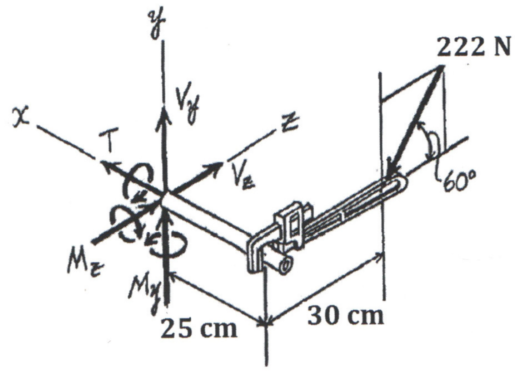
$$\sigma_x = \frac{M_y\,(19\ mm)}{I_y} = \frac{(27750\ Nmm)(25\ mm)}{306785.9\ mm^4} = 3.4\ MPa$$

$$\tau_{xy} = \frac{Tr_B}{J} = \frac{(57677.3\ Nmm)(25\ mm)}{408884.6\ mm^4} = 3.52\ MPa$$

$$\tau_{xz})_{Vz} = 0$$

$$\tau_{xy})_{Vy} = \frac{V_y Q_A}{I_z t} = \frac{(192.25\ N)(\frac{4\times 25}{3\pi}\frac{\pi}{2}25^2 - \frac{4\times 19}{3\pi}\frac{\pi}{2}19^2)}{(204442.3\ mm^4)((25-19)\times 2)} = 0.48\ MPa$$

$$\tau_{xy})A = 3.52 - 0.48 = 3.1\ MPa$$



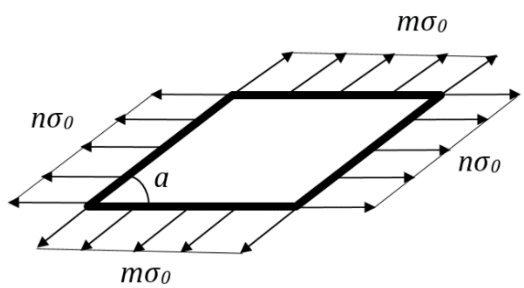
$$\sigma_{1,2})A = \frac{\sigma_x + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} = 5.53, -1.05\ MPa$$

$$\tau_{max})A = \left|\frac{\sigma_1 - \sigma_2}{2}\right| = 3.29\ \text{MPa}$$

$$\sigma_{1,2})B = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 5.23, -1.84\ MPa$$

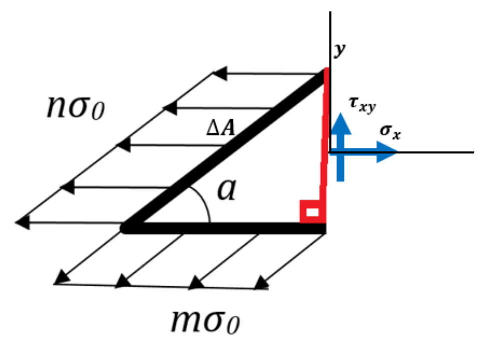
$$\tau_{max})\boldsymbol{B} = \left|\frac{\sigma_1 - \sigma_2}{2}\right| = 3.53\ \text{MPa}$$

TBR 5: Find principal stresses and maximal shear stress for the element shown (1393).



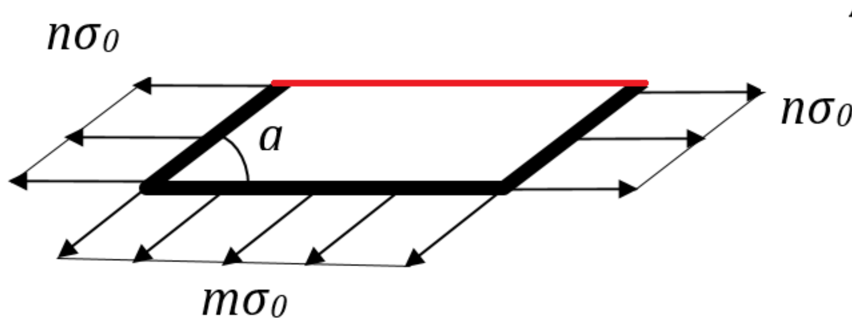
$$\sum F_x = 0 \rightarrow \sigma_x(\Delta A \sin\alpha) - n\sigma_0(\Delta A) - m\sigma_0 \cos\alpha\,(\Delta A\cos\alpha) = 0 \rightarrow \sigma_x \sin\alpha = (n + m\cos^2\alpha)\sigma_0$$

$$\sigma_x = \frac{(n + m\cos^2\alpha)\sigma_0}{\sin\alpha}$$



$$\sum F_y = 0 \rightarrow \tau_{xy}(\Delta A \sin\alpha) - m\sigma_0 \sin\alpha\,(\Delta A\cos\alpha) = 0$$

$$\rightarrow \tau_{xy} = m\sigma_0 \cos\alpha$$



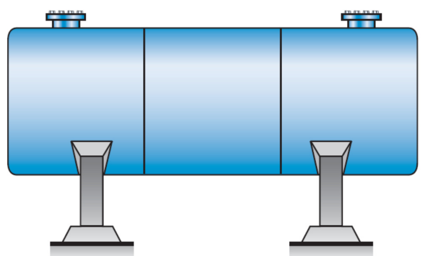
$$\rightarrow \sigma_y = m\sigma_0 \sin\alpha$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \checkmark$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \checkmark$$

Stresses at the *Outer Surface* of Thin-walled Pressure Vessels

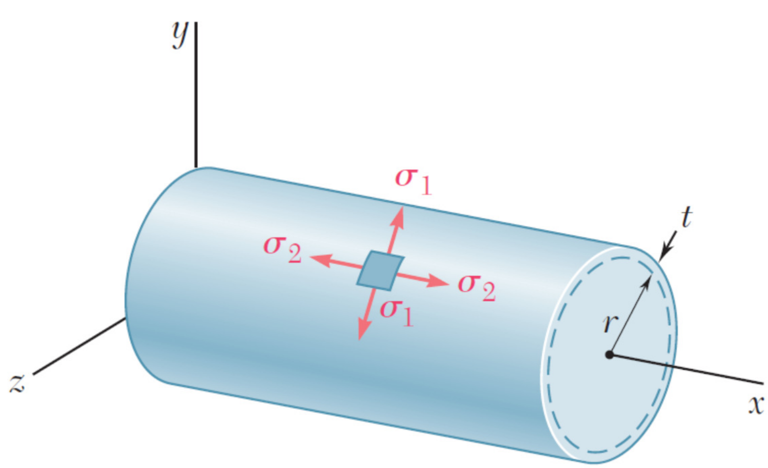
Thin-walled pressure vessels (with r/t ≥ 10) provide an important application of the analysis of plane stress. **Pressure vessels** are closed structures containing liquids or gases under pressure. Familiar examples include tanks, pipes, and pressurized cabins in aircraft and space vehicles. When pressure vessels have walls that are thin in comparison to their overall dimensions, they are included within a more general category known as **shell structures**. Other examples of shell structures are roof domes, airplane wings, and submarine hulls. In this section we consider thin-walled pressure vessels of cylindrical shape that are found in industrial settings (compressed air tanks and rocket motors), in homes (fire extinguishers and spray cans), and in the countryside (propane tanks and grain silos). Pressurized pipes, such as water-supply pipes and penstocks, are also classified as cylindrical pressure vessels.



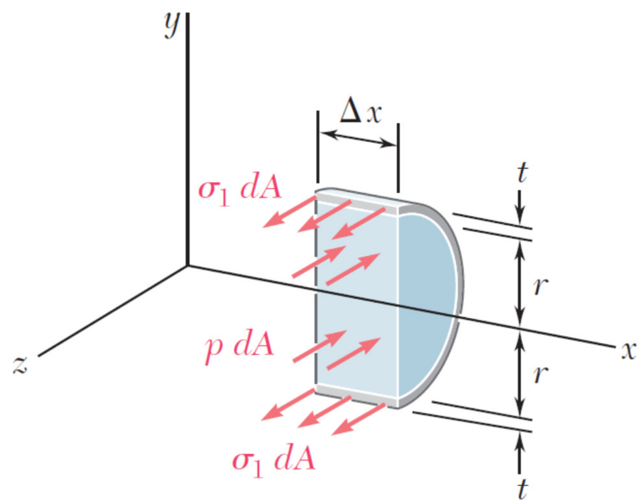
Aloha Airlines Flight 243

Examples of Spherical and Cylindrical Pressure Vessels

Consider a cylindrical vessel of inner radius r and wall thickness t containing a fluid under pressure. We aim to determine the stresses exerted on a small element of wall with sides respectively parallel and perpendicular to the axis of the cylinder. Because of the axisymmetry of the vessel and its contents, it is clear that no shearing stress is exerted on the element. The normal stresses σ_1 and σ_2, **which are almost constant throughout the thickness (varies by less than 5% from the inside of the vessel wall to the outside)**, shown in the Figure are therefore principal stresses. The stress σ_1 is known as the ***hoop*** or ***circumferential*** *stress* and the stress σ_2 is called the ***longitudinal*** *stress*.

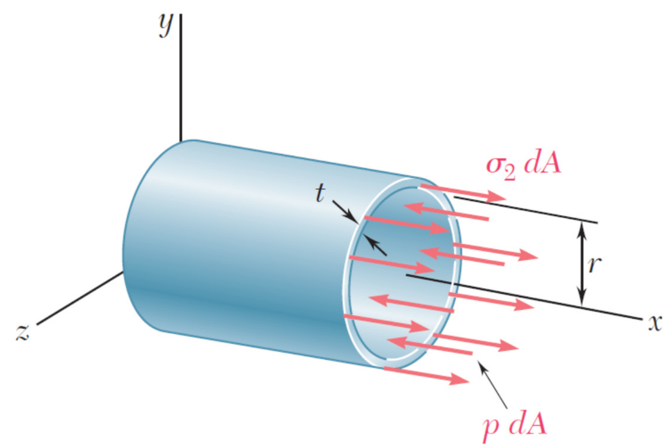


In order to determine the hoop stress σ_1, we detach a portion of the vessel and its contents bounded by the *xy* plane and by two planes parallel to the *yz* plane at a distance Δx from each other. The forces parallel to the *z* axis acting on the free body defined in this fashion consist of the elementary internal forces $\sigma_1\, dA$ on the wall sections, and of the elementary pressure forces $p\, dA$ exerted on the portion of fluid included in the free body. Note that p denotes ***the gage pressure*** of the fluid, i.e., the excess of the inside pressure over the outside atmospheric pressure. The resultant of the internal forces $\sigma_1\, dA$ is equal to the product of σ_1 and of the cross-sectional area $2t\,\Delta x$ of the wall, while the resultant of the pressure forces $p\, dA$ is equal to the product of p and of the area $2r\,\Delta x$. Writing the equilibrium equation $\Sigma F_z = 0$, we have:



$$p\,(2r\,\Delta x) - \sigma_1\,(2t\,\Delta x) = 0 \rightarrow \sigma_1 = \frac{pr}{t} = \sigma_\theta$$

To determine the longitudinal stress σ_2, we now pass a section perpendicular to the x axis and consider the free body consisting of the portion of the vessel and its contents located to the left of the section. The forces acting on this free body are the elementary internal forces $\sigma_2\, dA$ on the wall section and the elementary pressure forces $p\, dA$ exerted on the portion of fluid included in the free body. Noting that the area of the fluid section is πr^2 and that the area of the wall section can be obtained by multiplying the circumference $2\pi r$ of the cylinder by its wall thickness t, we write the equilibrium equation $\Sigma F_x = 0$:



$$\sigma_2\,(2\pi r t) - p(\pi r^2) = 0 \rightarrow \sigma_2 = \frac{pr}{2t} = \sigma_L$$

The maximum in-plane shearing stress is equal to:

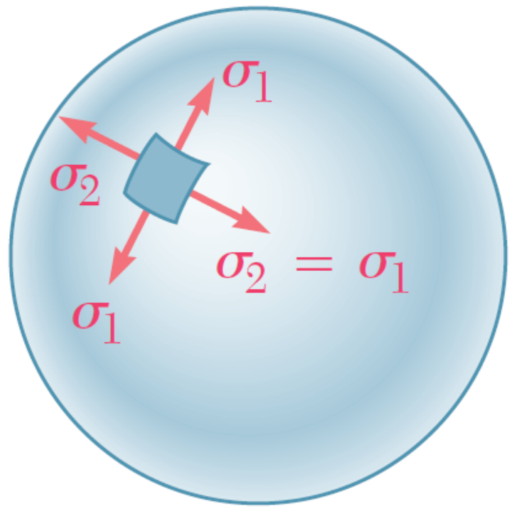
$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\frac{pr}{t} - \frac{pr}{2t}}{2} = \frac{pr}{4t}$$

This stress is exerted on an element obtained by rotating the original element through 45° within the plane tangent to the surface of the vessel. Note that to fill a pressure vessel with gas or liquid, or take the gas or liquid out, there must be a hole in the pressure-vessel wall and some sort of “connector”. The stress formulas developed above do not apply to stresses in the immediate vicinity of such discontinuities in the pressure-vessel wall (due to the stress concentration not considered here).

Thin-walled spherical pressure vessel used for storage of propane in this oil refinery

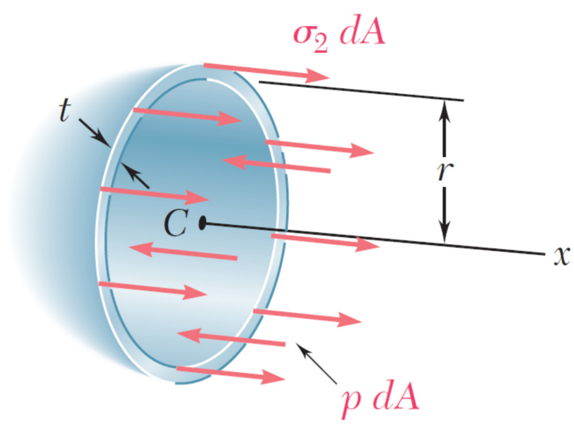
We now consider a spherical vessel of inner radius r and wall thickness t, containing a fluid under a gage pressure p. For reasons of symmetry, the stresses exerted on the four faces of a small element of wall must be equal, i.e., $\sigma_1 = \sigma_2$. To determine the value of the stress, we pass a section through the center C of the vessel and consider the free body consisting of the portion of the vessel and its contents located to the left of the section as shown. The equation of equilibrium for this free body is then considered:

$$\sigma_2\,(2\pi rt) - p(\pi r^2) = 0 \rightarrow \sigma_2 = \frac{pr}{2t} = \sigma_1$$



Pressurized spherical vessel

Since the principal stresses σ_1 and σ_2 are equal we conclude that the in-plane normal stress is constant and that the in-plane maximum shearing stress is zero.



Limitations of thin-shell theory:

1. The wall thickness must be small in comparison to the other dimensions (the ratio r/t should be 10 or more).

2. The internal pressure must exceed the external pressure (to avoid inward buckling).

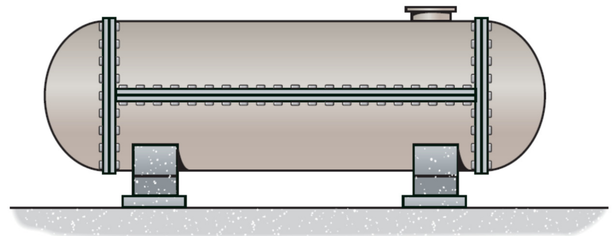
3. The analysis presented in this section is based only on the effects of internal pressure (the effects of external loads, reactions, the weight of the contents, and the weight of the structure are not considered).

4. The formulas derived in this section are valid throughout the wall of the vessel *except* near points of stress concentrations.

5. The plane-stress state is only valid at the outer surface of the vessel while at the inner surface in addition to above stresses we also have $\sigma_3 = -p$. Therefore the state of stress is three dimensional at the inner surface. This compressive stress varies from p at the inner surface to zero at the outer surface. As the magnitude of this stress is very small (its maximal value is p at the inner surface) compared to other stresses (hoop and longitudinal which are pr/t or $pr/2t$) we neglect σ_3 in our analysis and apply the foregoing equations for both inner and out surfaces of vessels.

*** Plot Mohr's Circle for both cylindrical and spherical vessels**

Example 4: The gas storage tank is fabricated by bolting together two half cylindrical thin shells and two hemispherical shells as shown. If the tank is designed to withstand a pressure of 3 MPa, determine the required minimum thickness of the cylindrical and hemispherical shells and the minimum required number of bolts for each hemispherical cap. The tank and the 25 mm diameter bolts are made from material having an allowable normal stress of 150 MPa and 250 MPa, respectively. The tank has an inner diameter of 4 m.



For the cylindrical portion of the tank, the hoop stress is twice as large as the longitudinal stress:

$$\sigma_{allow} = \frac{pr}{t} \rightarrow 150\ MPa = \frac{3\ MPa \times 2000\ mm}{t_C} \rightarrow t_C = 40\ mm$$

For the hemispherical portion:

$$\sigma_{\mathrm{allow}} = \frac{\mathrm{pr}}{2\mathrm{t}} \rightarrow 150\ \mathrm{MPa} = \frac{3\ \mathrm{MPa} \times 2000\ \mathrm{mm}}{2\mathrm{t_S}} \rightarrow \mathrm{t_S} = 20\ \mathrm{mm}$$

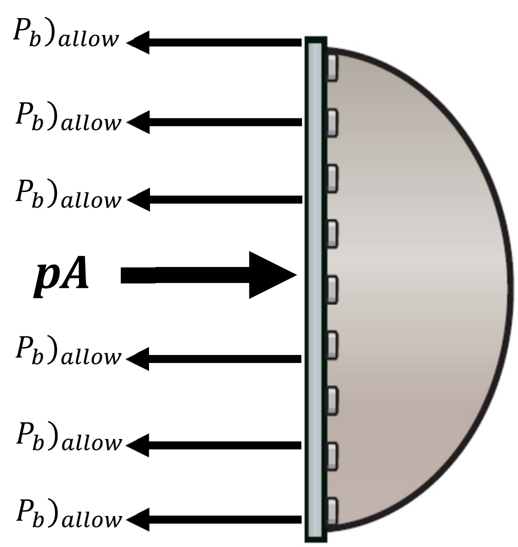
The allowable tensile force for each bolt is:

$$P_b)_{allow} = 250\ MPa \times \frac{\pi}{4}(25\ mm)^2 = 122718.46\ N$$

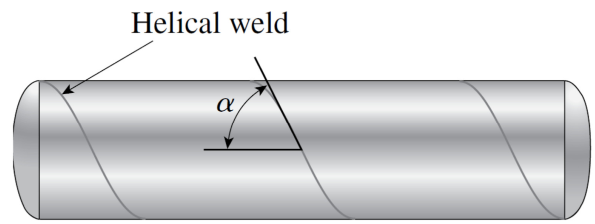
Consider the free-body diagram of the hemispherical cap:

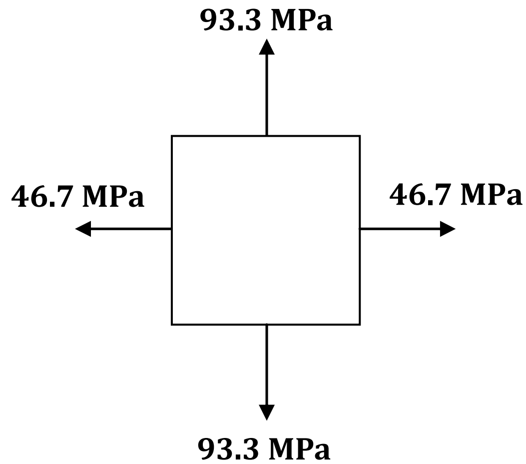
$$3\ MPa \times \pi\ (2000\ mm)^2 - n \times 122718.46\ N\ = 0$$

$$\rightarrow n = 307.2 \rightarrow n = 308\ bolts$$



Example 5: A pressurized steel tank is constructed with a helical weld that makes an angle a = 55° with the longitudinal axis (see figure). The tank has radius r = 0.6 m, wall thickness t = 18 mm, and internal pressure p = 2.8 MPa. Also, the steel has modulus of elasticity E = 200 GPa and Poisson's ratio v = 0.30. Determine the following quantities for the cylindrical part of the tank. (a) The circumferential and longitudinal stresses. (b) The maximum in-plane shear stresses. (c) The normal and shear stresses acting on planes parallel and perpendicular to the weld.



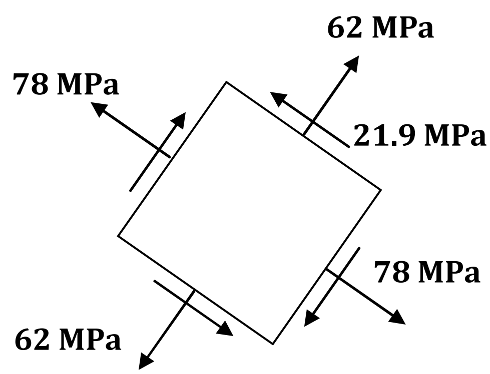


Hoop Stress:

$$\sigma_\theta = \frac{pr}{t} = \frac{(2.8\,MPa)(600\,mm)}{18\,mm} = 93.33\,MPa$$

Longitudinal Stress:

$$\sigma_L = \frac{pr}{2t} = \frac{(2.8\,MPa)(600\,mm)}{2 \times 18\,mm} = 46.7\,MPa$$



Maximum In-Plane Shear Stress:

$$\tau_{max} = \frac{\sigma_\theta - \sigma_L}{2} = \frac{93.33 - 46.7}{2} = 23.3\,MPa$$

Stress on Weld:

$$\theta = 90° - 55° = 35°$$

$$\sigma_x = 46.7\,MPa, \sigma_y = 93.3\,MPa, \tau_{xy} = 0$$

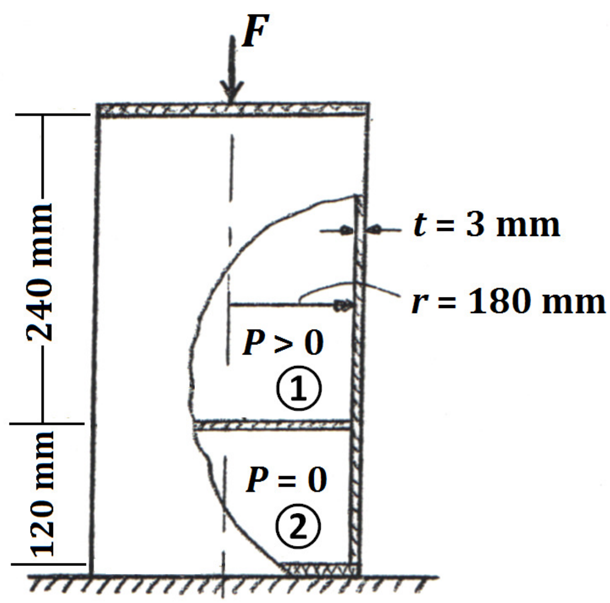
$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\,sin\,2\theta = \frac{46.7 + 93.3}{2} + \frac{46.7 - 93.3}{2}\cos 70° = 62\,MPa$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\,sin\,2\theta = \frac{46.7 + 93.3}{2} - \frac{46.7 - 93.3}{2}\cos 70° = 78\,MPa$$

$$\tau_{x'y'} = -\frac{(\sigma_x - \sigma_y)}{2}sin2\theta + \tau_{xy}\,\cos 2\theta = 21.9\,MPa$$

Check that: $\boldsymbol{\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}}$

TBR 6: For the system shown find the maximum pressure P and axial compressive force F that can be applied if the allowable stress is 90 MPa.



Part ①:

$$\sigma_\theta = \frac{Pr}{t}$$

$$\rightarrow P_{max} = \frac{\sigma_\theta t}{r} = \frac{90\ MPa \times 3\ mm}{180\ mm} = 1.5\ MPa$$

$$\sigma_L = \frac{Pr}{2t} - \frac{F}{2\pi rt} = \pm 90\ MPa$$

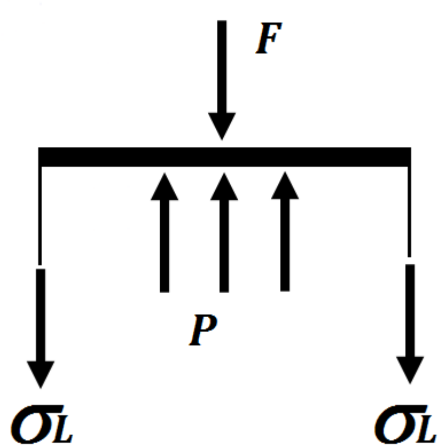
$$\rightarrow F_{max} = 2\pi rt\left(\frac{Pr}{2t} \pm 90\ MPa\right) = 2\pi rt\left(\frac{1.5\ MPa\ \times 180\ mm}{2 \times 3\ mm} \pm 90\ MPa\right)$$
$$= 458\ kN, -152.7\ kN$$

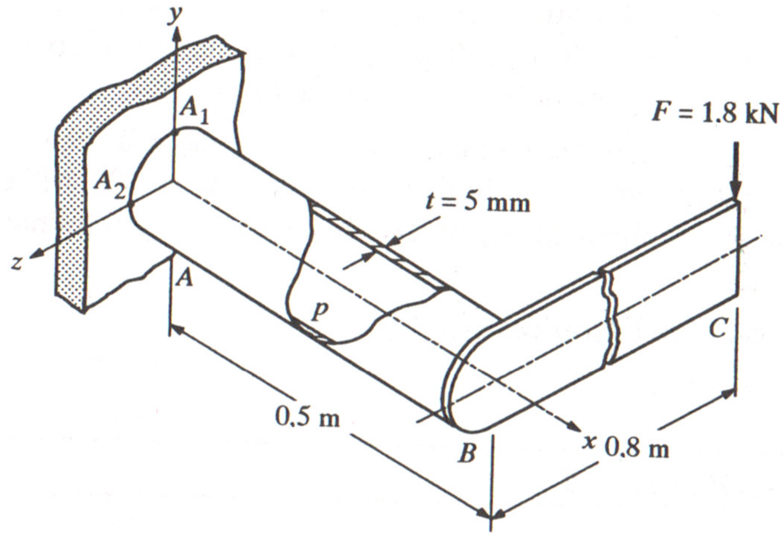
Part ②:

$$\sigma_\theta = 0$$

$$\sigma_L = -\frac{F}{2\pi rt} = \pm 90\ MPa \rightarrow F_{max} = 305.4\ kN, -305{,}4\ kN$$

Conclusion: $P_{max} = 1.5\ MPa \quad and \quad F_{max} = 305.4\ kN$





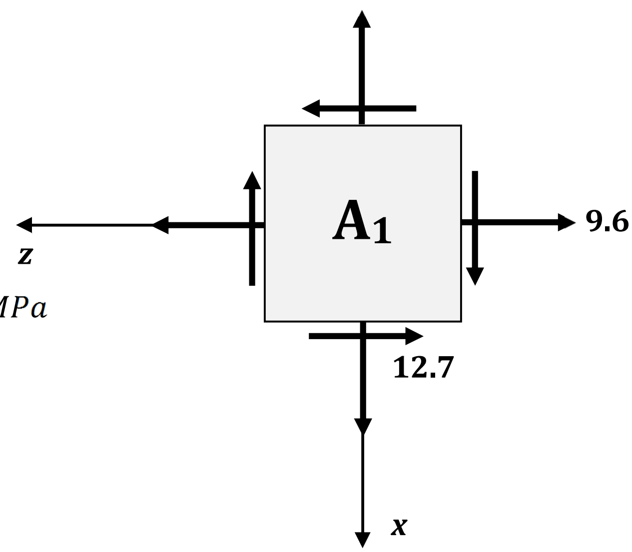
Example 6: For the system shown, determine principal stresses at points A_1 and A_2 (internal pressure p of the cylinder is equal to 0.8 MPa and its radius to mid-thickness is equal to 60 mm).

Hoop and Longitudinal stresses due to p:

$$\sigma_\theta = \frac{Pr}{t} = \frac{(0.8\,MPa)(60\,mm)}{5\,mm} = 9.6\,MPa$$

$$\sigma_L = \frac{Pr}{2t} = \frac{(0.8\,MPa)(60\,mm)}{2 \times 5\,mm} = 4.8\,MPa$$

Shear stress due to torsion:



$$\tau_T = \frac{Tr}{J}\ or\ \frac{T}{2\bar{A}t} = \frac{(1.8 \times 10^3 N \times 800\,mm)}{2 \times \pi 60^2 mm^2 \times 5\,mm} = 12.7\,MPa$$

Normal stress due to bending at A_1:

$$\sigma_b = \frac{Mc_{A1}}{I} = \frac{(1.8 \times 10^3 N \times 500\,mm)(60\,mm)}{\frac{\pi}{4}(62.5^4 - 57.5^4)\,mm^4} = 15.9\,MPa$$

Shear stress due to shear loading at A_1:

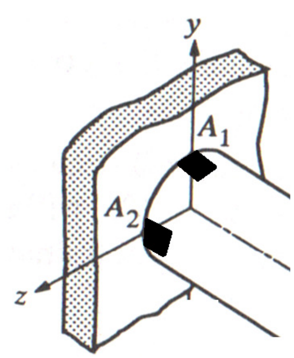
$$\tau_V = \frac{VQ}{It} = 0$$

Resultant stresses:

$$\sigma_x = \sigma_L + \sigma_b = 4.8 + 15.9 = 20.7\,MPa, \sigma_z = \sigma_\theta = 9.6\,MPa, \tau_{xz} = \tau_T = -12.7\,MPa$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} = \frac{20.7 + 9.6}{2} \pm \sqrt{\left(\frac{20.7 - 9.6}{2}\right)^2 + (-12.7)^2} = 29.03\,MPa, 1.27\,MPa$$

Normal stress due to bending at A_2: $\sigma_b = \frac{Mc_{A2}}{I} = 0$



Shear stress due to shear loading at A_2:

$$\tau_V = \frac{VQ}{IT} = \frac{(1.8 \times 10^3 N)\left(\frac{4r_2}{3\pi}\frac{\pi r_2^2}{2} - \frac{4r_1}{3\pi}\frac{\pi r_1^2}{2}\right) mm^3}{\frac{\pi}{4}(62.5^4 - 57.5^4)\,mm^4 \times 10\,mm} = 1.91\,MPa$$

Resultant stresses: $\sigma_x = \sigma_L + \sigma_b = 4.8 + 0 = 4.8\,MPa,\ \ \sigma_y = \sigma_\theta = 9.6\,MPa$

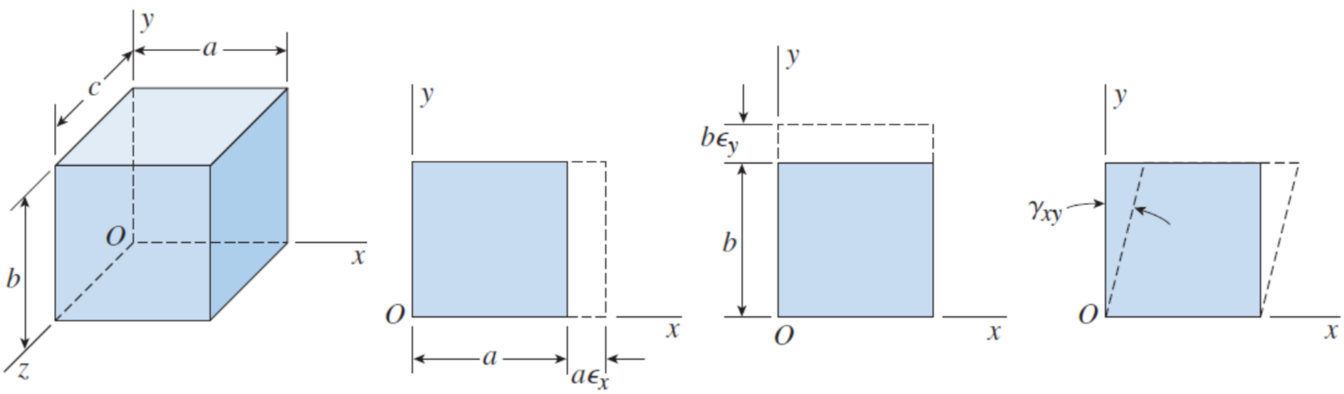
$$\tau_{xy} = 12.7\ - 1.9 = 10.8\,MPa \rightarrow \sigma_{1,2} = \frac{\sigma_x + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} = 18.3\,MPa, -3.9\,MPa$$

Transformation of Plane Strain

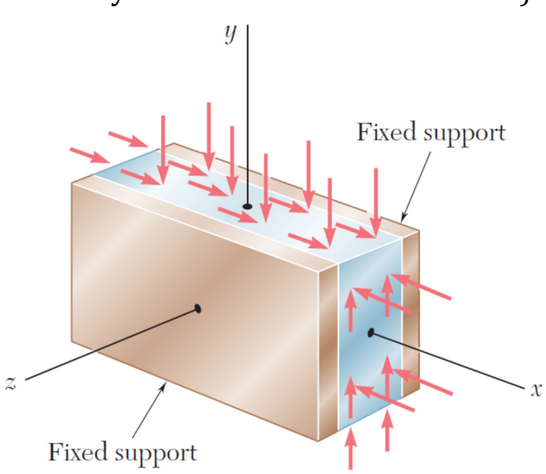
The strains at a point in a loaded structure vary according to the orientation of the axes, in a manner similar to that for stresses. In this section we will derive the transformation equations that relate the strains in inclined directions to the strains in the reference directions. These transformation equations are widely used in laboratory and field investigations involving measurements of strains.

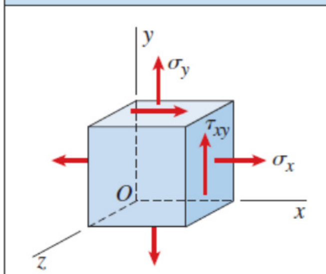
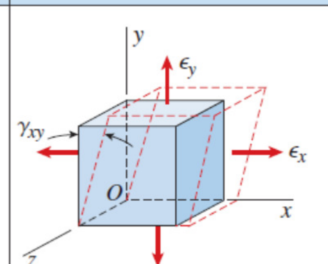
Plane Strain versus Plane Stress

Let us begin by explaining what is meant by plane strain and how it relates to plane stress. Consider a small element of material having sides of lengths *a*, *b*, and *c* in the *x*, *y*, and *z* directions, respectively. If the only deformations are those in the *xy* plane, then three strain components may exist; the normal strain ε_x in the *x* direction, the normal strain ε_y in the *y* direction, and the shear strain γ_{xy}. An element of material subjected to these strains (and *only* these strains) is said to be in a state of **plane strain**. It follows that an element in plane strain has no normal strain ε_z in the *z* direction and no shear strains γ_{xz} and γ_{yz} in the *xz* and *yz* planes, respectively. Thus, plane strain is defined by the following conditions: $\varepsilon_z = 0$, $\gamma_{xz} = 0$ and $\gamma_{yz} = 0$. The remaining strains (ε_x, ε_y, and γ_{xy}) may have nonzero values.



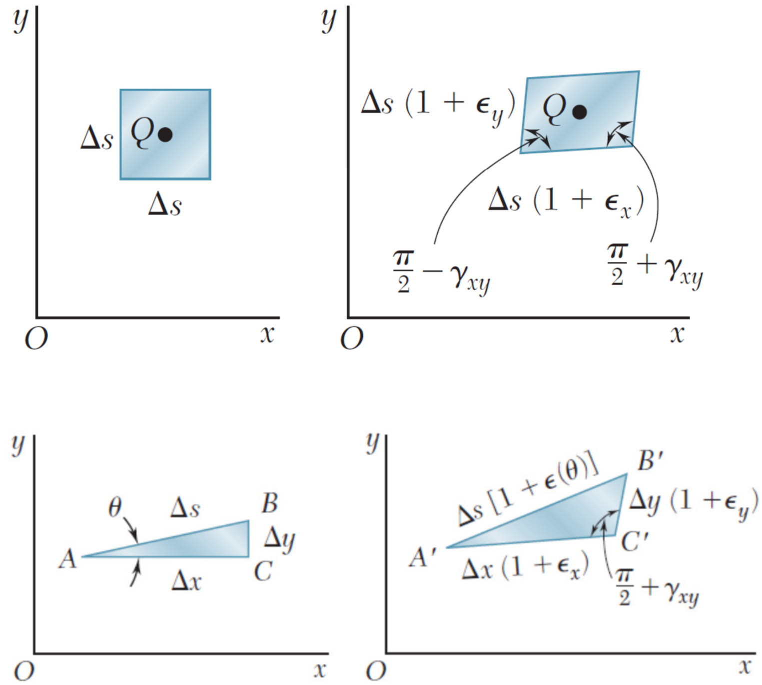
From the preceding definition, we see that plane strain occurs when the front and rear faces of an element of material are fully restrained against displacement in the z direction as shown below (another example of plane strain is a long bar subjected to uniformly distributed transverse loads).



	Plane stress	Plane strain
		
Stresses	$\sigma_z = 0$ $\tau_{xz} = 0$ $\tau_{yz} = 0$ σ_x, σ_y, and τ_{xy} may have nonzero values	$\tau_{xz} = 0$ $\tau_{yz} = 0$ σ_x, σ_y, σ_z, and τ_{xy} may have nonzero values
Strains	$\gamma_{xz} = 0$ $\gamma_{yz} = 0$ ϵ_x, ϵ_y, ϵ_z, and γ_{xy} may have nonzero values	$\epsilon_z = 0$ $\gamma_{xz} = 0$ $\gamma_{yz} = 0$ ϵ_x, ϵ_y, and γ_{xy} may have nonzero values

Transformation Equations for Plane Strain

Let us assume that a state of plane strain exists at point Q (with $\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0$), and that it is defined by the strain components ε_x, ε_y, and γ_{xy} associated with the x and y axes. As we know, this means that a square element of center Q, with sides of length Δs respectively parallel to the x and y axes, is deformed into a parallelogram with sides of length respectively equal to $\Delta s\ (1 + \varepsilon_x)$ and $\Delta s\ (1 + \varepsilon_y)$, forming angles of $\pi/2 - \gamma_{xy}$ and $\pi/2 + \gamma_{xy}$ with each other. We first derive an expression for the normal strain $\varepsilon(\Theta)$ along a line AB forming an arbitrary angle Θ with the x axis. To do so, we consider the right triangle ABC and the oblique triangle $A'B'C'$ into which triangle ABC is deformed.



$$(A'B')^2 = (A'C')^2 + (B'C')^2 - 2(A'C')(B'C')\ \cos\left(\frac{\pi}{2} + \gamma_{xy}\right)$$

$$(\Delta s)^2\big(1+\varepsilon(\theta)\big)^2 = (\Delta x)^2(1+\varepsilon_x)^2 + (\Delta y)^2\big(1+\varepsilon_y\big)^2 - 2(\Delta x)(1+\varepsilon_x)(\Delta y)\big(1+\varepsilon_y\big)\cos\left(\frac{\pi}{2}+\gamma_{xy}\right)$$

$\Delta x = \Delta s\ cos\theta \quad \Delta y = \Delta s\ sin\theta \quad \cos\left(\frac{\pi}{2}+\gamma_{xy}\right) = -\sin\gamma_{xy} = -\gamma_{xy}$, By neglecting 2nd order terms:

$$\varepsilon(\theta) = \varepsilon_x\cos^2\theta + \varepsilon_y\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta \qquad \varepsilon(45°) = \frac{1}{2}\big(\varepsilon_x+\varepsilon_y+\gamma_{xy}\big) \rightarrow \gamma_{xy} = 2\varepsilon(45°) - \big(\varepsilon_x+\varepsilon_y\big)$$

$$\varepsilon_{x'} = \frac{\varepsilon_x+\varepsilon_y}{2} + \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x+\varepsilon_y}{2} - \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon(\theta+45°) = \frac{\varepsilon_x+\varepsilon_y}{2} - \frac{\varepsilon_x-\varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

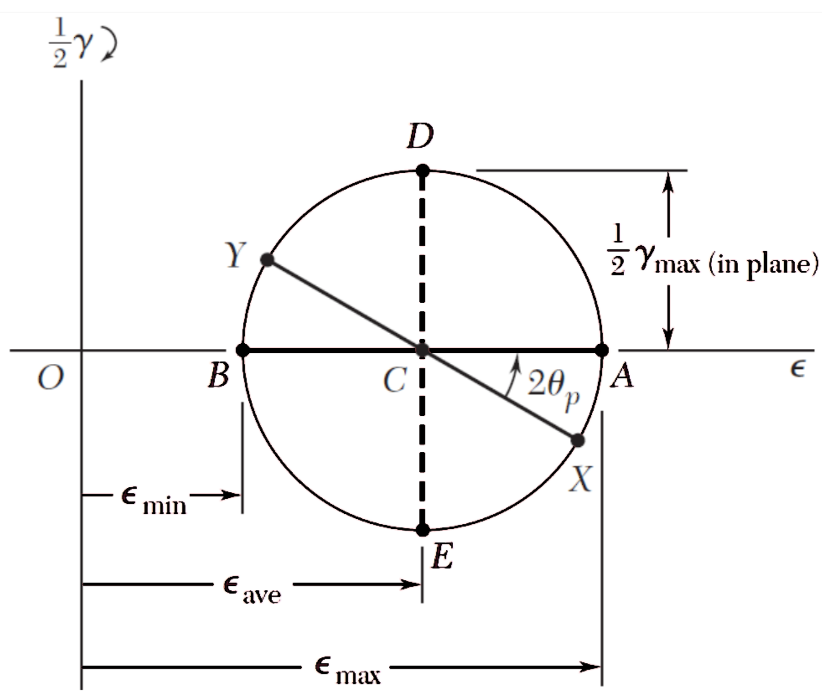
$$\gamma_{x'y'} = 2\varepsilon(\theta+45°) - \big(\varepsilon_{x'}+\varepsilon_{y'}\big)$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x-\varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

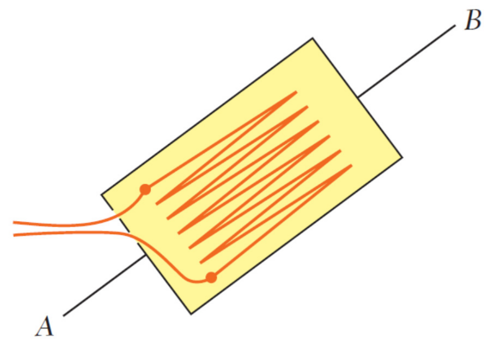
$$\varepsilon_{avg} = \frac{\varepsilon_x+\varepsilon_y}{2}, R = \sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\tan 2\theta_P = \frac{\gamma_{xy}}{\varepsilon_x-\varepsilon_y},\ \varepsilon_{max,min} = \varepsilon_{avg} \pm R$$

$$\gamma_{max} = 2R = \sqrt{\big(\varepsilon_x-\varepsilon_y\big)^2 + \big(\gamma_{xy}\big)^2}$$

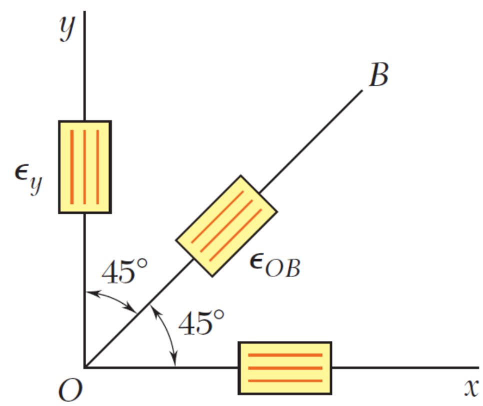


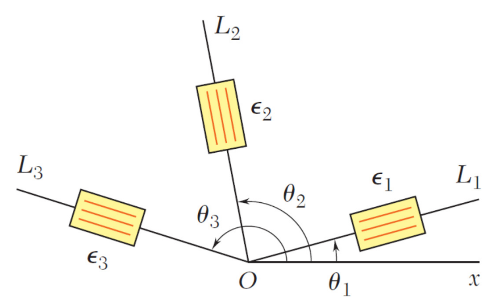
Measurements of Strain; Strain Rosette



Electrical strain gage.

The normal strain can be determined in any given direction on the surface of a structural element or machine component by scribing two gage marks A and B across a line drawn in the desired direction and measuring the length of the segment AB before and after the load has been applied. If L is the undeformed length of AB and δ its deformation, the normal strain along AB is $\varepsilon_{AB} = \delta/L$. A more convenient and more accurate method for the measurement of normal strains is provided by electrical strain gages. A typical electrical strain gage consists of a length of thin wire arranged as shown and cemented to two pieces of paper. In order to measure the strain ε_{AB} of a given material in the direction AB, the gage is cemented to the surface of the material, with the wire folds running parallel to AB. As the material elongates, the wire increases in length and decreases in diameter, causing the electrical resistance of the gage to increase. By measuring the current passing through a properly calibrated gage, the strain ε_{AB} can be determined accurately and continuously as the load is increased. The strain components ε_x and ε_y can be determined at a given point of the free surface of a material by simply measuring the normal strain along x and y axes drawn through that point. We note that a third measurement of normal strain, made along the bisector OB of the angle formed by the x and y axes, enables us to determine the shearing strain γ_{xy} as well $\gamma_{xy} = 2\varepsilon(45°) - (\varepsilon_x + \varepsilon_y)$. It should be noted that the strain components ε_x, ε_y, and γ_{xy} at a given point could be obtained from normal strain measurements made along *any three lines* drawn through that point. Denoting respectively by Θ_1, Θ_2, and Θ_3 the angle each of the three lines forms with the x axis, by ε_1, ε_2, and ε_3 the corresponding strain measurements we write the three equations:





Strain rosette.

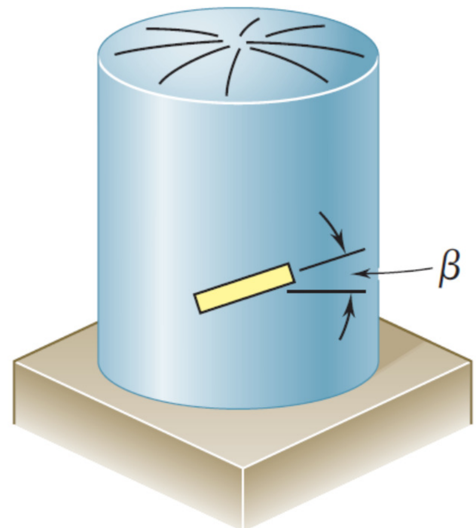
$$\varepsilon_1 = \varepsilon_x \cos^2\theta_1 + \varepsilon_y \sin^2\theta_1 + \gamma_{xy}\sin\theta_1\cos\theta_1$$

$$\varepsilon_2 = \varepsilon_x \cos^2\theta_2 + \varepsilon_y \sin^2\theta_2 + \gamma_{xy}\sin\theta_2\cos\theta_2$$

$$\varepsilon_3 = \varepsilon_x \cos^2\theta_3 + \varepsilon_y \sin^2\theta_3 + \gamma_{xy}\sin\theta_3\cos\theta_3$$

which can be solved simultaneously for ε_x, ε_y, and γ_{xy}. The arrangement of strain gages used to measure the three normal strains ε_1, ε_2, and ε_3 is known as a *strain rosette*. Also stresses σ_x, σ_y, and τ_{xy} can be determined through Hook's law.

Example 7: A single strain gage forming an angle β = 18° with a horizontal plane is used to determine the gage pressure in the cylindrical steel tank shown. The cylindrical wall of the tank is 6-mm thick, has a 600-mm inside diameter, and is made of a steel with E = 200 GPa and ν = 0.30. Determine the pressure in the tank indicated by a strain gage reading of 280μ.

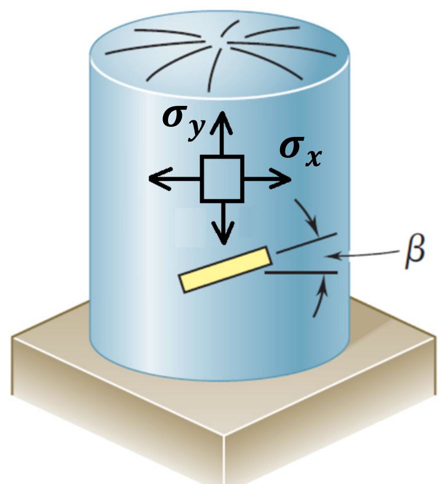


$$\sigma_x = \frac{pr}{t}, \sigma_y = \frac{pr}{2t}, \sigma_z = 0$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} = \frac{pr}{tE} - \frac{\nu pr}{2tE} = 2.125 \times 10^{-4}\, p$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_z}{E} = \frac{pr}{2tE} - \frac{\nu pr}{tE} = 5.0 \times 10^{-5}\, p$$

$$\varepsilon_{18^\circ} = \varepsilon_x \cos^2(18^\circ) + \varepsilon_y \sin^2(18^\circ) + \gamma_{xy} \sin(18^\circ)\cos(18^\circ) = 280 \times 10^{-6}$$

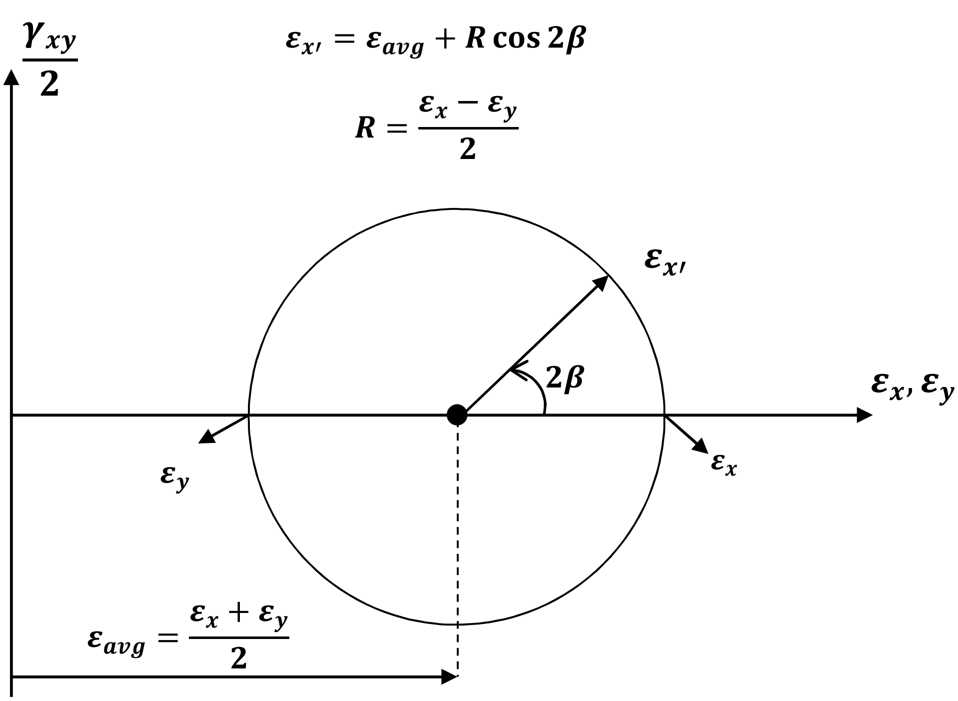


$$\gamma_{xy} = 0 \quad as \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

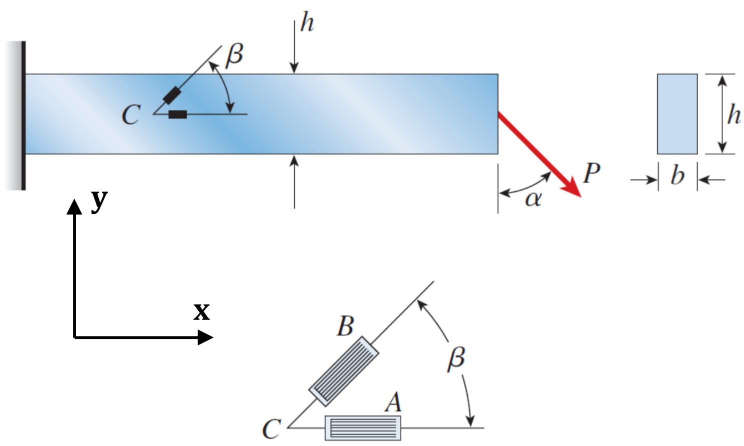
$$\rightarrow \varepsilon_x \cos^2(18^\circ) + \varepsilon_y \sin^2(18^\circ) = 280 \times 10^{-6}$$

$$\rightarrow\ 2.125 \times 10^{-4}\, p\ \cos^2(18^\circ) + 5.0 \times 10^{-5}\, p\ \sin^2(18^\circ) = 280 \times 10^{-6}$$

$$\rightarrow p = 1.42\ MPa$$



Example 8: A cantilever beam of rectangular cross section (width b = 25 mm, height h = 100 mm) is loaded by a force P that acts at the mid-height of the beam and is inclined at an angle α to the vertical (see figure). Two strain gages are placed at point C, which also is at the mid-height of the beam. Gage A measures the strain in the horizontal direction and gage B measures the strain at an angle β = 60° to the horizontal. The measured strains are $\varepsilon_a = 125 \times 10^{-6}$ and $\varepsilon_b = -375 \times 10^{-6}$. Determine the force P and the angle α, assuming the material is steel with E = 200 GPa and ν = 1/3.



At point C:

Axial Force: $F = P\sin\alpha$

Shear Force: $V = P\cos\alpha$

At the neutral axis no bending stress is produced.

$$\sigma_x = \frac{F}{A} = \frac{P\sin\alpha}{bh}, \quad \sigma_y = 0, \tau_{xy} = \frac{VQ}{It} = -\frac{3V}{2A} = -\frac{3P\cos\alpha}{2bh}$$

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{P\sin\alpha}{bhE} = 125 \times 10^{-6} \rightarrow P\sin\alpha = 62\,500\ N \quad (1)$$

$$\varepsilon_y = -\upsilon\frac{\sigma_x}{E} = -41.67 \times 10^{-6}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = -\frac{3P\cos\alpha}{2bhG} \xrightarrow{G=\frac{E}{2(1+\upsilon)}} \gamma_{xy} = -8 \times 10^{-9}\,P\cos\alpha$$

$For\ \theta = 60°$:

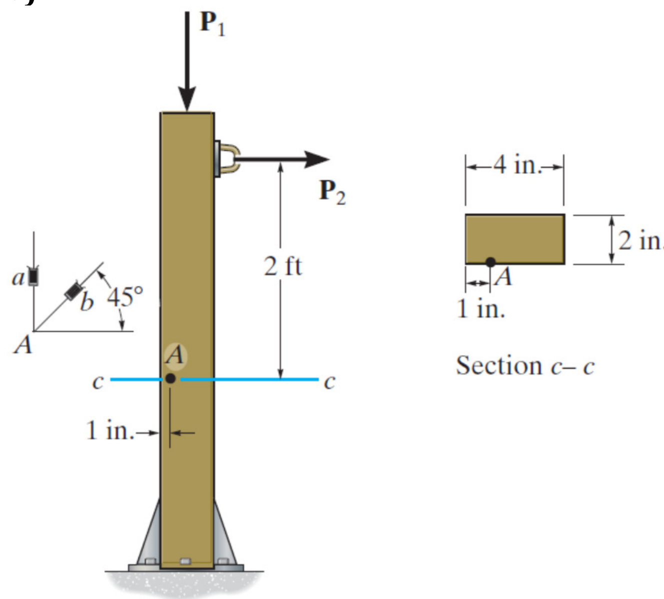
$$\varepsilon_{x'} = \varepsilon_x\cos^2(60°) + \varepsilon_y\sin^2(60°) + \gamma_{xy}\sin(60°)\cos(60°) = \varepsilon_b = -375 \times 10^{-6}$$

$$\rightarrow 125 \times 10^{-6}\cos^2(60°) - 41.67 \times 10^{-6}\sin^2(60°) - 8 \times 10^{-9}\,P\cos\alpha\sin(60°)\cos(60°) = -375 \times 10^{-6}$$

$P\cos\alpha = 108\,206\ N$ (2)

$$\rightarrow (1)\ and\ (2): \alpha = 30°, P = 125\ kN$$

TBR **7:** The A-36 steel post $(\upsilon = 0.32, G = 11 \times 10^3 \frac{lb}{in^2} = 11\, ksi, E = 29\, ksi)$ is subjected to the forces shown. If the strain gauges a and b at point A give readings of $\varepsilon_a = 300 \times 10^{-6}$ and $\varepsilon_b = 175 \times 10^{-6}$ determine the magnitudes of $\mathbf{P}_1$ and $\mathbf{P}_2$.



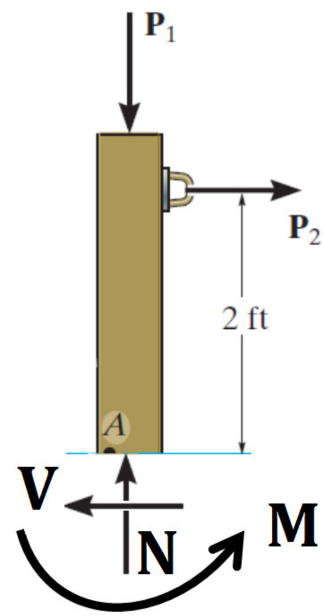
$$\sum F_x = 0 \rightarrow V = P_2$$

$$\sum F_y = 0 \rightarrow N = P_1$$

$$\sum M = 0 \rightarrow M = 2P_2$$

Normal Stress at A:

$$\sigma_A = \frac{N}{A} + \frac{Mc}{I} = -\frac{P_1}{4 \times 2} + \frac{(2 \times 12\, P_2)(1)}{\frac{1}{12} 2 \times (4)^3} = -0.125 P_1 + 2.25 P_2$$



$$\textit{Shear Stress at A}: \tau_A = \frac{VQ_A}{It} = \frac{P_2 \times (1.5 \times 1 \times 2)}{\frac{1}{12} 2 \times (4)^3 \times 2} = 0.14\, P_2$$

$$\varepsilon_a = \varepsilon_x \cos^2\theta_a + \varepsilon_y \sin^2\theta_a + \gamma_{xy} \sin\theta_a \cos\theta_a$$

$$\rightarrow 300 \times 10^{-6} = \varepsilon_x \cos^2 90° + \varepsilon_y \sin^2 90° + \gamma_{xy} \sin 90° \cos 90°$$

$$\rightarrow \varepsilon_y = 300 \times 10^{-6} \ (1)$$

$$\varepsilon_b = \varepsilon_x \cos^2\theta_b + \varepsilon_y \sin^2\theta_b + \gamma_{xy} \sin\theta_b \cos\theta_b$$

$$\rightarrow 175 \times 10^{-6} = \varepsilon_x \cos^2 45° + \varepsilon_y \sin^2 45° + \gamma_{xy} \sin 45° \cos 45°$$

$$\rightarrow \varepsilon_x + \gamma_{xy} = 50 \times 10^{-6} \ (2)$$

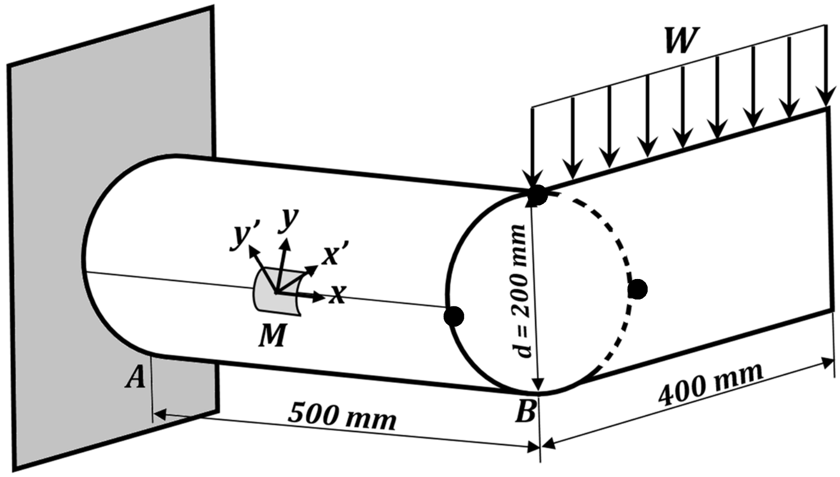
$$\varepsilon_x = \frac{\sigma_x}{E} - \upsilon \frac{\sigma_y}{E} - \upsilon \frac{\sigma_z}{E} \xrightarrow{\sigma_x = \sigma_z = 0} \varepsilon_x = -\upsilon \frac{\sigma_y}{E}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \upsilon \frac{\sigma_x}{E} - \upsilon \frac{\sigma_z}{E} \rightarrow \varepsilon_y = \frac{\sigma_y}{E} = \varepsilon_x / -\upsilon \rightarrow \textit{from}\ (1): \varepsilon_x = -96 \times 10^{-6} \rightarrow \textit{from}\ (2): \gamma_{xy} = 146 \times 10^{-6}$$

$$\tau_{xy} = G\gamma_{xy} \rightarrow 0.14\, P_2 = (11 \times 10^3)(146 \times 10^{-6}) \rightarrow P_2 = 11.42\, kip$$

$$\varepsilon_y = \frac{\sigma_y}{E} \rightarrow 300 \times 10^{-6} = \frac{-0.125 P_1 + 2.25 P_2}{29 \times 10^3} \rightarrow P_1 = 136\, kip$$

TBR 8: Strain in the element M (middle of AB on the outter surface) is equal to 80µ and -80µ in x' and y' directions, respectively (x' and y' are at 45 degrees with respect to x and y). Determine magnitude of *W*. Based on the calculated W, find the magnitude and location of the maximal normal stress in the cylinder (E = 210 GPa, G = 85 GPa) (1392).



As element M is located on the neutral axis of the section, no normal stress occurs in the element. So we have: $\varepsilon_x = \varepsilon_y = 0$.

$$\varepsilon'_x = \varepsilon_x \cos^2 45° + \varepsilon_y \sin^2 45° + \gamma_{xy} \sin 45° \cos 45° = 80 \times 10^{-6} \rightarrow \gamma_{xy} = 160 \times 10^{-6}$$

Or

$$\varepsilon'_y = \varepsilon_x \cos^2 135° + \varepsilon_y \sin^2 135° + \gamma_{xy} \sin 135° \cos 135° = -80 \times 10^{-6} \rightarrow \gamma_{xy} = 160 \times 10^{-6}$$

$$\tau_{xy} = G\gamma_{xy} = 85\ GPa \times 160 \times 10^{-6} \rightarrow \tau_{xy} = 13.6\ MPa$$

$$for\ element\ M: \tau_{xy} = \frac{VQ}{It} + \frac{Tr}{J} = -\frac{4}{3}\frac{V}{A} + \frac{Tr}{J} = -\frac{4}{3}\frac{400w}{\pi(100)^2} + \frac{400w \times 200 \times 100}{\frac{\pi}{2}(100)^4}$$

$$= 13.6\ MPa$$

$$w = 400.5\ N/mm$$

Maximal normal stress occurs at point N where the bending moment is maximum:

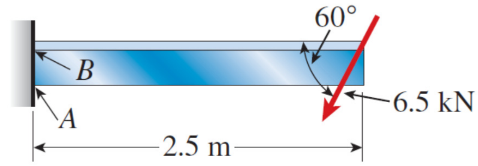
$$\sigma_x = \frac{MC}{I} = \frac{(400\ mm \times 400.5\ N/mm \times 500\ mm) \times 100\ mm}{\frac{\pi}{4}(100)^4\ mm\hat{}4} = 102\ MPa$$

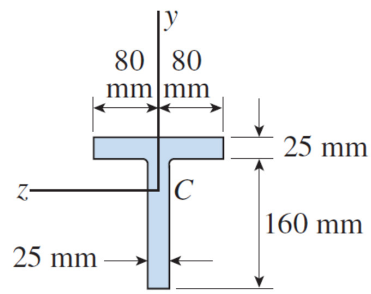
$$\tau_{xy} = \frac{VQ}{It} + \frac{Tr}{J} = 0 + \frac{Tr}{J} = \frac{400\ mm \times 400.5\ N/mm \times 200\ mm \times 100\ mm}{\frac{\pi}{2}(100)^4\ mm\hat{}4} = 20.4\ MPa$$

$$\sigma_{max,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{102 + 0}{2} \pm \sqrt{\left(\frac{102 - 0}{2}\right)^2 + 20.4^2}$$

$$\sigma_{max} = 105.9\ MPa$$

CHAPTER 8: Combined Loadings





Example 1: A cantilever beam of T-section is loaded by an inclined force of magnitude 6.5 kN (see figure). The line of action of the force is inclined at an angle of 60° to the horizontal and intersects the top of the beam at the end cross section. The beam is 2.5 m long and the cross section has the dimensions shown. Determine the principal stresses and the maximum shear stress at points *A* and *B* in the web of the beam near the support.

$P = 6.5\text{ kN}$ $\quad L = 2.5\text{ m}$ $\quad A = 2(160\text{ mm})(25\text{ mm})$

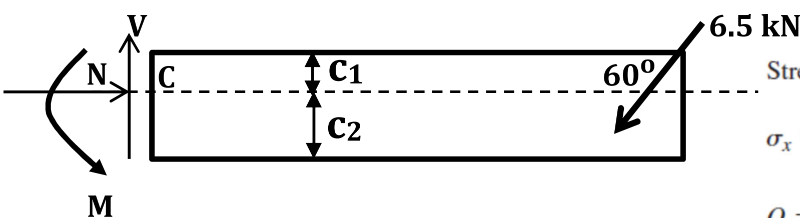
$A = 8 \times 10^3\text{ mm}^2$ $\quad b = 160\text{ mm}$ $\quad t = 25\text{ mm}$

Location of centroid C $\quad$ From Eq. (12-7b) in Chapter 12:

$$c_2 = \frac{\Sigma(y_i A_i)}{A} \qquad c_2 = \frac{(160\text{ mm})(25\text{ mm})\left(160 + \frac{25}{2}\right)\text{mm} + (160\text{ mm})(25\text{ mm})(80\text{ mm})}{A}$$

$c_2 = 126.25\text{ mm}$ $\quad c_1 = 185\text{ mm} - c_2$ $\quad c_1 = 58.75\text{ mm}$

$$I_Z = \frac{1}{12}25(160)^3 + 25 \times 160 \times (126.25 - 80)^2 + \frac{1}{12}160(25)^3 + 160 \times 25 \times \left(160 + \frac{25}{2} - 126.25\right)^2 = 2.585 \times 10^7 mm^4$$



$$\sum F_x = 0 \rightarrow N_0 = 3.25\ kN$$

$$\sum F_y = 0 \rightarrow V = 5.63\ kN$$

$$\sum M_C = 0 \rightarrow M = 13.88 \times 10^6\ Nmm$$

Stress at point B (top of web)

$$\sigma_x = \frac{N_0}{A} - \frac{M(c_1 - t)}{I_z} \qquad \sigma_x = 17.715\text{ MPa} \qquad \sigma_y = 0$$

$$Q = bt\left(c_1 - \frac{t}{2}\right) \qquad Q = 1.85 \times 10^5\text{ mm}^3$$

$$\tau_{xy} = -\frac{VQ}{I_z t} \qquad \tau_{xy} = -1.611\text{ MPa}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_1 = 17.86\text{ MPa}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_2 = -0.145\text{ M}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + {\tau_{xy}}^2} \qquad \tau_{max} = 9.00\text{ MPa} \quad \leftarrow$$

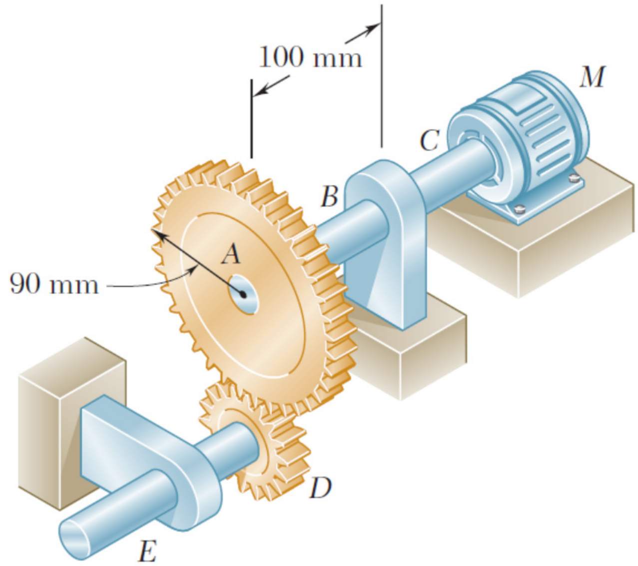
Stress at point A (bottom of web)

$$\sigma_x = \frac{N_0}{A} + \frac{Mc_2}{I_Z} \qquad \sigma_x = \text{-68.19 MPa} \qquad \sigma_y = 0 \qquad \tau_{xy} = 0$$

Uniaxial stress: $\sigma_1 = \sigma_y \qquad \sigma_2 = \sigma_x \qquad \tau_{max} = \left|\frac{\sigma_x}{2}\right|$

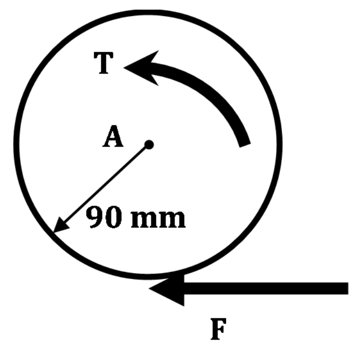
$\sigma_1 = 0 \quad \leftarrow \qquad \sigma_2 = \text{-68.19 MPa} \quad \leftarrow \qquad \tau_{max} = 34.1\text{ MPa} \quad \leftarrow$

Example 2: The solid shaft *ABC* and the gears shown are used to transmit 10 kW from the motor *M* to a machine tool connected to gear *D*. Knowing that the motor rotates at 240 rpm and that τ_{all} = 60 MPa, determine the smallest permissible diameter of shaft *ABC*.



$$P = Tw \rightarrow T = \frac{10\,000\,W}{240 \times \frac{2\pi}{60}\frac{rad}{s}} = 397.887\,Nm$$

$$\sum T_A = 0 \rightarrow T - r_A F = 0 \rightarrow F = \frac{397\,887\,Nmm}{90\,mm} = 4421\,N$$

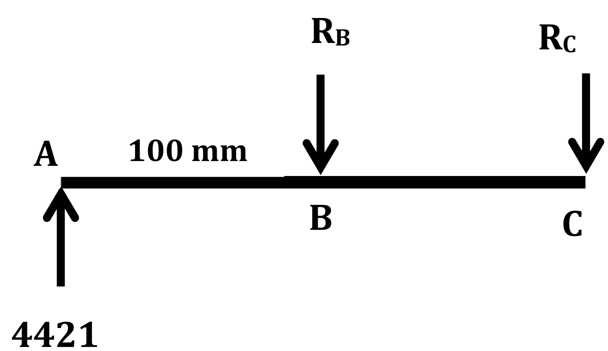


$$M_{max} = M_B = 4421\,N \times 100\,mm = 442\,100\,Nmm = M$$

$$\sigma_B = \frac{Mc}{I} = \frac{Mr}{I}$$

$$\tau_B = \frac{Tr}{J}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{\frac{Mr}{I}}{2}\right)^2 + \left(\frac{Tr}{J}\right)^2}$$



$$\tau_{max} = \sqrt{\left(\frac{Mr}{J}\right)^2 + \left(\frac{Tr}{J}\right)^2} = \frac{r}{J}\sqrt{M^2 + T^2} = \frac{r}{\frac{\pi}{2}r^4}\sqrt{M^2 + T^2}$$

$$\tau_{max} = \frac{2}{\pi r^3}\sqrt{M^2 + T^2} \rightarrow r^3 = \frac{2\sqrt{M^2 + T^2}}{\pi\,\tau_{max}}$$

$$r = \left(\frac{2\sqrt{M^2 + T^2}}{\pi\,\tau_{max}}\right)^{\frac{1}{3}} = \left(\frac{2\sqrt{442100^2 + 397887^2}}{\pi \times 60}\right)^{\frac{1}{3}} \rightarrow r = 18.48\,mm$$

$$d \cong 37\,mm$$