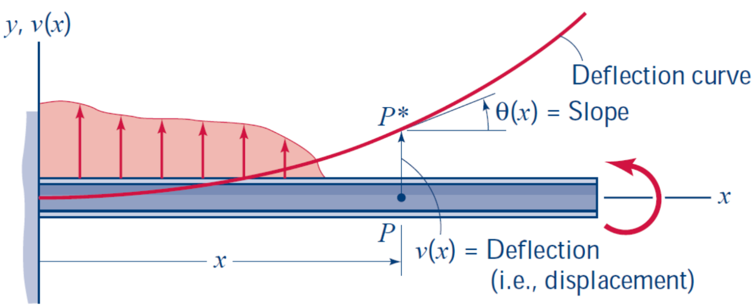
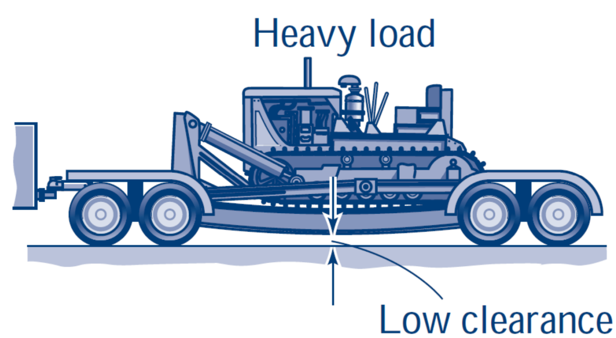
CHAPTER 9: Deflection and Slope of Beams

Introduction: When a beam with a straight longitudinal axis is loaded by lateral forces (and/or moments), the axis is deformed into a curve, called the deflection curve of the beam. In Chapter 4, we used the curvature of the bent beam to determine the normal strains and stresses in the beam. However, we did not develop a method for finding the deflection curve itself. In this chapter, we will determine the equation of the deflection curve and also find deflections at specific points along the axis of the beam. As indicated in the following figure, the **deflection curve** is characterized by a function $v(x)$ that gives the **transverse displacement** (i.e., displacement in the y direction) of the points that lie along the axis of the beam. The **slope** of the deflection curve is labeled $\theta(x)$. Deflections are sometimes calculated in order to verify that they are within tolerable limits. For instance, specifications for the design of buildings usually place upper limits on the deflections. Large deflections in buildings are unsightly (and even unnerving) and can cause cracks in ceilings and walls. In the design of machines and aircraft, specifications may limit deflections in order to prevent undesirable vibrations. For example, the beams of the equipment trailer shown below must not deflect so much under load that the clearance between the trailer and the ground becomes unacceptably small. Also of interest is that knowledge of the deflections is required to analyze *indeterminate beams*.

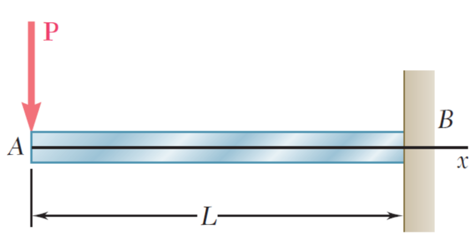


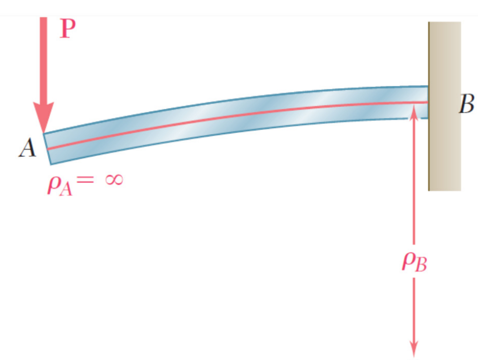


Three methods are introduced in this chapter to find deflection of beams:

(1) Integration (use of differential equations of the deflection curve) method,
(2) Singularity function method,
(3) Superposition method.
(4) Energy Methods (Will be studied in “Strength of Materials II”)

Integration Method (differential equations of the deflection curve)





$$\frac{\mathbf{1}}{\boldsymbol{\rho(x)}} = \frac{\boldsymbol{M(x)}}{\boldsymbol{EI}} = \frac{\boldsymbol{-Px}}{\boldsymbol{EI}}$$

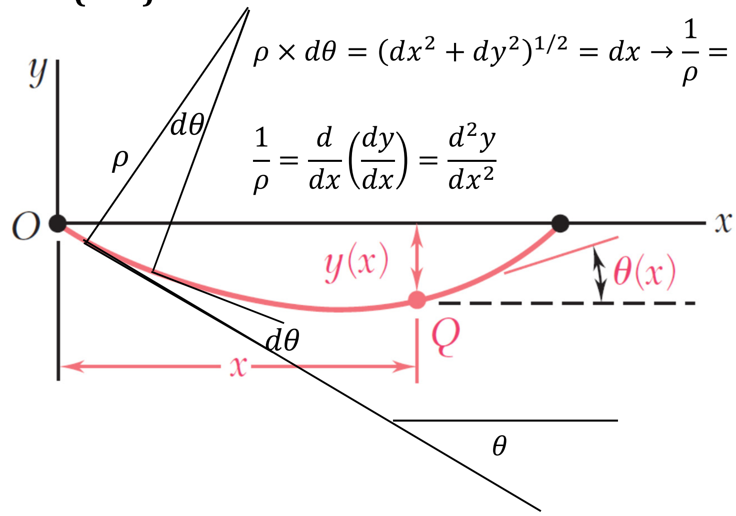
$$\rightarrow \frac{1}{\rho_A} = \frac{0}{EI} \rightarrow \rho_A = \infty \quad and \quad \frac{1}{\rho_B} = \frac{-PL}{EI} \rightarrow |\rho_B| = \frac{EI}{PL}$$

From elementary calculus we know that the curvature of a plane curve at a point $Q(x,y)$ of the curve can be expressed as:

$$\frac{1}{\rho(x)}=\frac{\dfrac{d^2y}{dx^2}}{\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$$

In the case of the elastic curve of a beam, the slope dy/dx is very small, and its square is negligible compared to unity:

$$\frac{1}{\rho(x)}=\frac{d^2y}{dx^2}\rightarrow\frac{d^2y}{dx^2}=\frac{M(x)}{EI}$$



$$\rho\times d\theta=(dx^2+dy^2)^{1/2}=dx\rightarrow\frac{1}{\rho}=\frac{d\theta}{dx}$$

$$\frac{1}{\rho}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$$

$$EI\frac{d^2y}{dx^2}=M(x)\rightarrow EI\frac{dy}{dx}=\int_0^x M(x)dx+C_1,\qquad \frac{dy}{dx}=\tan\theta\cong\theta(x)$$

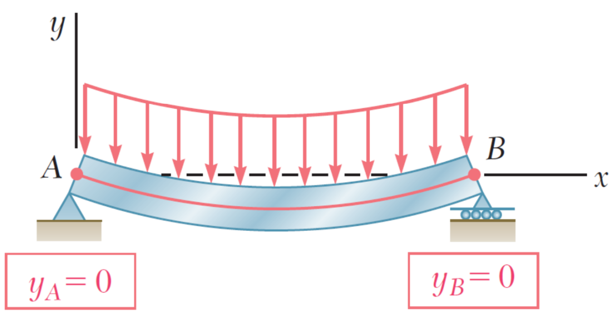
$$EI\theta(x)=\int_0^x M(x)dx+C_1$$

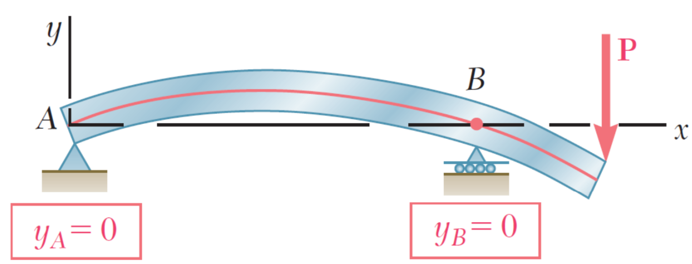
1- Deformations due to shear loading are neglected

2- y is the deflection curve of the neutral axis

$$\rightarrow EIy=\int_0^x\left(\int_0^x M(x)dx\right)dx+C_1x+C_2\quad (C_1\ and\ C_2\ are\ clculated\ from\ boundary\ conditions)$$

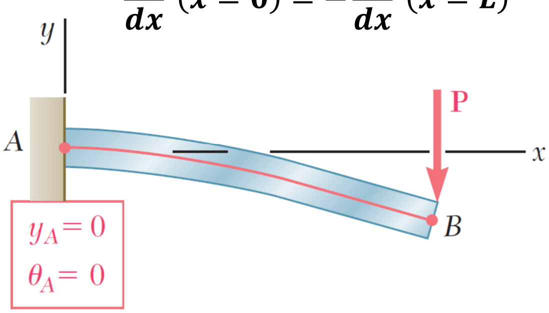
Boundary/ Continuity/Symmetry Conditions:

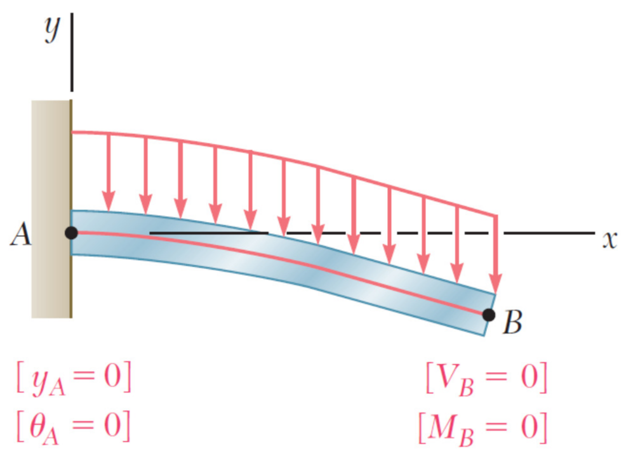




Symmetry Conditions: $\frac{dy}{dx}\left(x=\frac{L}{2}\right)=0$

$$\frac{dy}{dx}(x=0)=-\frac{dy}{dx}(x=L)$$

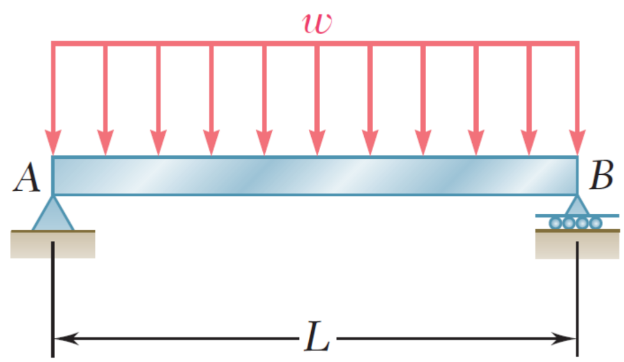




Direct determination of the elastic curve from the load distribution:

$$\frac{d^2y}{dx^2}=\frac{M(x)}{EI}\rightarrow\frac{d^3y}{dx^3}=\frac{V(x)}{EI}\rightarrow\frac{d^4y}{dx^4}=\frac{-w(x)}{EI}\rightarrow Four\ times\ integration\ and\ four\ boundary\ conditions$$

Example 1: Determine the equation of the elastic curve and the maximum deflection of the beam.

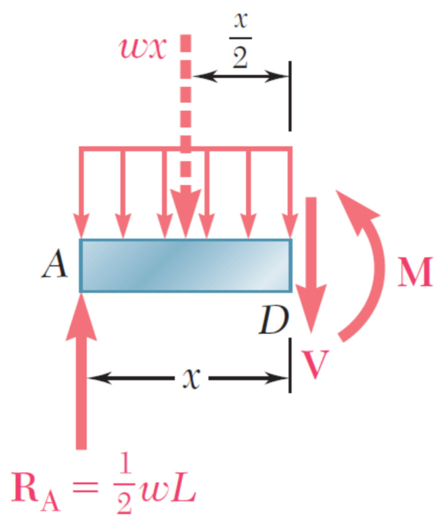


From Statics: $R_A = R_B = \frac{wL}{2}$

Elastic curve:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

We need to calculate M(x) From Statics:



$$M(x) = R_A x - wx \times \frac{x}{2} = \frac{wLx}{2} - \frac{wx^2}{2}$$
$$= \frac{w}{2}(-x^2 + Lx)$$

$$\rightarrow EI\frac{d^2y}{dx^2} = \frac{w}{2}(-x^2 + Lx)$$

$$\rightarrow EI\frac{dy}{dx} = \frac{w}{2}\left(-\frac{x^3}{3} + \frac{Lx^2}{2}\right) + C_1$$

$$\rightarrow EIy = \frac{w}{2}\left(-\frac{x^4}{12} + \frac{Lx^3}{6}\right) + C_1 x + C_2$$

Boundary Conditions: we have two unknowns (C_1 & C_2) so we need two boundary conditions:

$(\mathbf{1})\ y(x = 0) = 0 \rightarrow C_2 = 0 \qquad (\mathbf{2})\ y(x = L) = 0 \ \rightarrow C_1 = -\frac{wL^3}{24}$

Or we can use **Symmetry Conditions:** $\frac{dy}{dx}\left(x = \frac{L}{2}\right) = 0.\ \ or\ \ \boldsymbol{\frac{dy}{dx}\,(x = 0) = -\frac{dy}{dx}\,(x = L)} \rightarrow C_1 = -\frac{wL^3}{24}$

$$\rightarrow y = \frac{w}{2EI}\left(-\frac{x^4}{12} + \frac{Lx^3}{6}\right) - \frac{wL^3}{24EI}x$$

Direct determination of the elastic curve from the load distribution:

$$\frac{d^4y}{dx^4} = \frac{-w(x)}{EI} \rightarrow EI\frac{d^4y}{dx^4} = -w \rightarrow EI\frac{d^3y}{dx^3} = -wx + C_1 \rightarrow EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2} + C_1 x + C_2$$
$$\rightarrow EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1\frac{x^2}{2} + C_2 x + C_3 \rightarrow EIy = -\frac{wx^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4$$

Four boundary conditions are required:

$(\mathbf{1})\ y(x = 0) = 0 \rightarrow C_4 = 0$

$(\mathbf{2})\ y(x = L)\ \rightarrow\ 0 = -\frac{wL^4}{24} + C_1\frac{L^3}{6} + 0 + C_3 L \rightarrow C_3 = -\frac{wL^3}{24}$

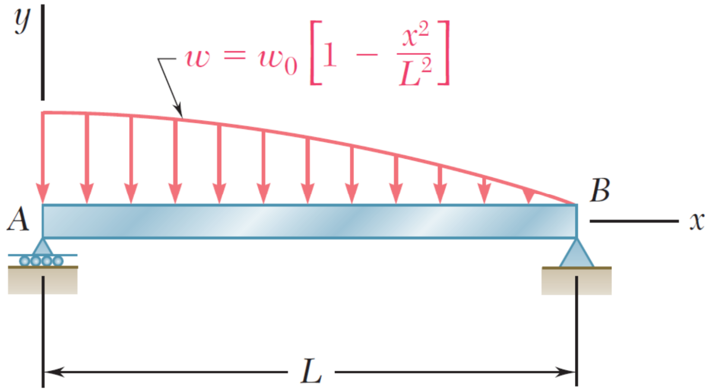
$(\mathbf{3})\ M(x = 0) = 0 \rightarrow M(x) = EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2} + C_1 x + C_2 \rightarrow 0 = 0 + 0 + C_2 \rightarrow C_2 = 0$

$(\mathbf{4})\ M(x = L) = 0 \rightarrow M(x) = EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2} + C_1 x + C_2 \rightarrow 0 = -\frac{wL^2}{2} + C_1 L \rightarrow C_1 = \frac{wL}{2}$

OR: $(\mathbf{5})\ \frac{dy}{dx}\left(x = \frac{L}{2}\right) = 0$

Advantage: No need to determine reaction forces, **Disadvantage:** More constants to be determined

Example 2: Determine the equation of the elastic curve and the deflection of the beam at midpoint.



$$\frac{d^4y}{dx^4}=\frac{-w(x)}{EI}\rightarrow EI\frac{d^4y}{dx^4}=-w_0\left(1-\frac{x^2}{L^2}\right)$$

$$\rightarrow EI\frac{d^3y}{dx^3}=-w_0\left(x-\frac{x^3}{3L^2}\right)+C_1$$

$$\rightarrow EI\frac{d^2y}{dx^2}=-w_0\left(\frac{x^2}{2}-\frac{x^4}{12L^2}\right)+C_1x+C_2$$

$$\rightarrow EI\frac{dy}{dx}=-w_0\left(\frac{x^3}{6}-\frac{x^5}{60L^2}\right)+C_1\frac{x^2}{2}+C_2x+C_3$$

$$\rightarrow EIy=-w_0\left(\frac{x^4}{24}-\frac{x^6}{360L^2}\right)+C_1\frac{x^3}{6}+C_2\frac{x^2}{2}+C_3x+C_4$$

Four Boundary conditions are required:

$(\mathbf{1})\ y(x=0)=0\rightarrow C_4=0$

$(\mathbf{2})\ y(x=L)=0\ \rightarrow\ 0=-w_0\left(\frac{L^4}{24}-\frac{L^6}{360L^2}\right)+C_1\frac{L^3}{6}+C_3L\rightarrow C_3=-\frac{11w_0L^3}{360}$

$(\mathbf{3})\ M(x=0)=0\rightarrow C_2=0$

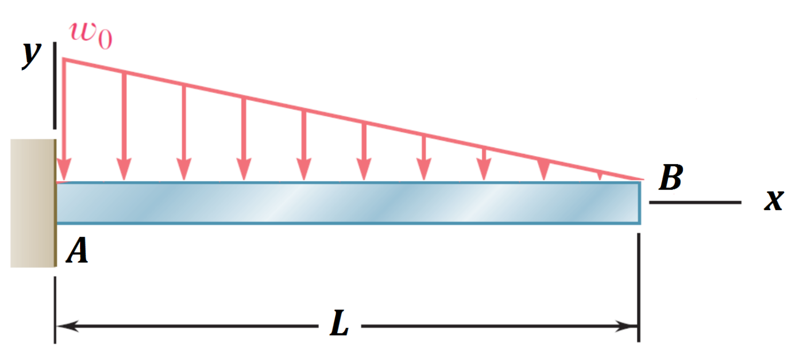
$(\mathbf{4})\ M(x=L)=0\rightarrow 0=-w_0\left(\frac{L^2}{2}-\frac{L^4}{12L^2}\right)+C_1L\rightarrow C_1=\frac{5w_0L}{12}$

$$y=\frac{w_0}{360EIL^2}(x^6-15L^2x^4+25L^3x^3-11L^5x)\rightarrow y_{x=\frac{L}{2}}=\frac{0.00916w_0L^4}{EI}\downarrow$$

Find slope at A:

$$EI\frac{dy}{dx}_{x=0}=C_3\rightarrow\theta=-\frac{11w_0L^3}{360EI}=\frac{11w_0L^3}{360EI}$$

TBR 1: Determine the equation of the elastic curve and the maximum deflection of the beam.



Considering x from right (easier way):

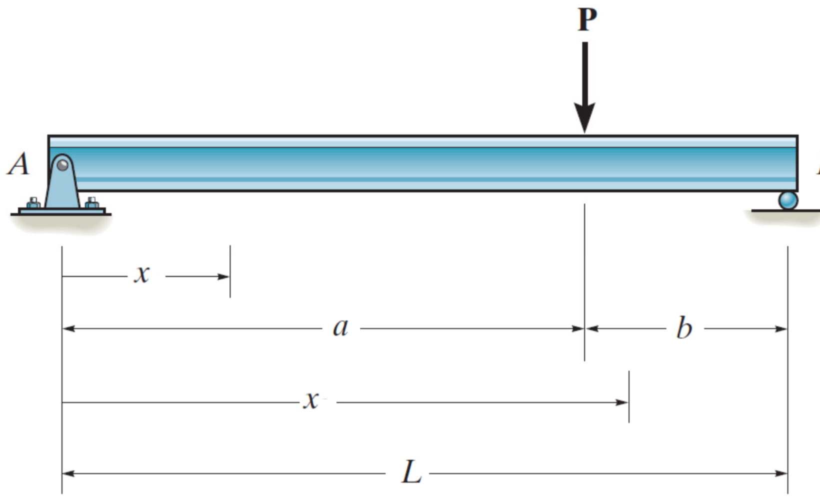
$$M(x)=\frac{-w_0}{2L}xx\times\frac{x}{3}\rightarrow EI\frac{d^2y}{dx^2}=\frac{-w_0x^3}{6L}$$

$$EI\frac{dy}{dx}=\frac{-w_0x^4}{24L}+C_1\rightarrow EIy=\frac{-w_0x^5}{120L}+C_1x+C_2\ \ Boundary\ cond.:y(x=L)=0,\frac{dy}{dx}(x=L)=0$$

$$C_1=\frac{w_0L^3}{24},C_2=-\frac{w_0L^4}{30}\rightarrow y=\frac{w_0}{120EIL}(-x^5+5L^4x-4L^5)\ \ (x\ must\ be\ considered\ from\ right)$$

Considering x from left, reactions at A must be calculated: $y=\frac{w_0}{120EIL}(x^5-5Lx^4+10L^2x^3-10L^3x^2)$

Example 4: Determine the equations of the elastic curve. EI is constant.



From Statics:

$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

$0 \le x \le a$

$$M(x) = \frac{Pb}{L}x$$

$$\frac{d^2y_1}{dx^2} = \frac{M(x)}{EI} \rightarrow EI\frac{d^2y_1}{dx^2} = \frac{Pb}{L}x$$

$$\rightarrow EI\frac{dy_1}{dx} = \frac{Pb}{L}\frac{x^2}{2} + C_1 \rightarrow EIy_1 = \frac{Pb}{L}\frac{x^3}{6} + C_1x + C_2$$

$a \le x \le L$

$$M(x) = \frac{Pb}{L}x - P(x-a) = Pa(1-\frac{x}{L})$$

$$\frac{d^2y_2}{dx^2} = \frac{M(x)}{EI} \rightarrow EI\frac{d^2y_2}{dx^2} = Pa\left(1-\frac{x}{L}\right) \rightarrow EI\frac{dy_2}{dx} = Pa\left(x-\frac{x^2}{2L}\right) + C_3$$

$$\rightarrow EIy_2 = Pa\left(\frac{x^2}{2}-\frac{x^3}{6L}\right) + C_3x + C_4$$

Boundary/Continuity Conditions:

(**1**) $y_1(x=0) = 0$
(**2**) $y_2(x=L) = 0$
(**3**) $y_1(x=a) = y_2(x=a)$
(**4**) $\frac{dy_1}{dx}(x=a) = \frac{dy_2}{dx}(x=a)$

$$C_1 = -\frac{Pb}{6L}(L^2-b^2)$$

$$C_2 = 0$$

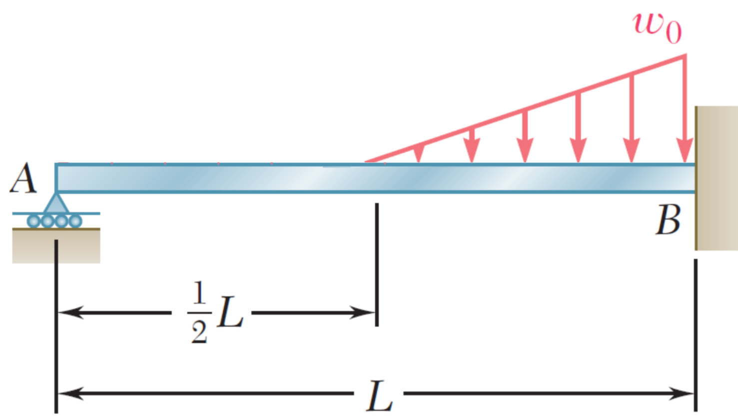
$$C_3 = -\frac{Pa}{6L}(2L^2+a^2)$$

$$C_4 = \frac{Pa^3}{6}$$

$$EIy_1 = \frac{Pb}{6L}x^3 - \frac{Pb}{6L}(L^2-b^2)x \quad 0 \le x \le a$$

$$EIy_2 = Pa\left(\frac{x^2}{2}-\frac{x^3}{6L}\right) - \frac{Pa}{6L}(2L^2+a^2)x + \frac{Pa^3}{6} \quad a \le x \le L$$

Example 5: Determine the equations of the elastic curve. *EI* is constant.



There are three unknowns (R_A, R_B, and M_B) and we only have two equilibrium equations being ΣF=0 and ΣM=0. So the problem is statically indeterminate (order 1) and we should first find reactions:

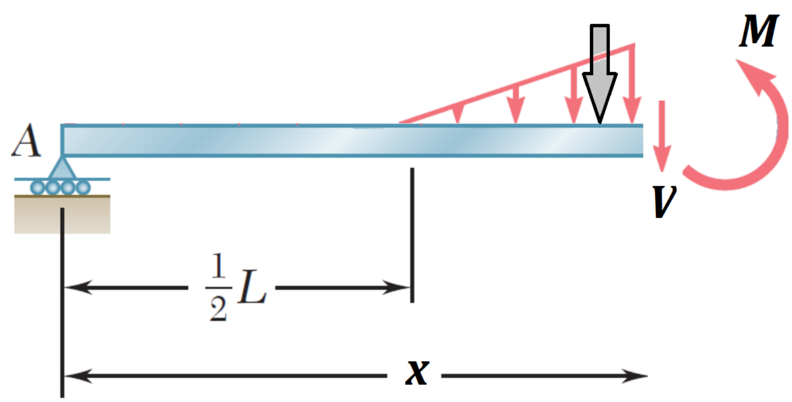
$\mathbf{0 \le x \le \frac{L}{2}}$

$M(x) = R_A x$

$\frac{d^2y_1}{dx^2} = \frac{M(x)}{EI} \to EI\frac{d^2y_1}{dx^2} = R_A x$

$\to EI\frac{dy_1}{dx} = R_A\frac{x^2}{2} + C_1$

$\to EIy_1 = R_A\frac{x^3}{6} + C_1x + C_2$



$\mathbf{\frac{L}{2} \le x \le L}$

$M(x) = R_A x - \frac{1}{2}\frac{2w_0}{L}(x-\frac{L}{2})(x-\frac{L}{2})\frac{(x-\frac{L}{2})}{3}$

$M(x) = R_A x - \frac{w_0}{3L}\left(x-\frac{L}{2}\right)^3$

$\frac{d^2y_2}{dx^2} = \frac{M(x)}{EI} \to EI\frac{d^2y_2}{dx^2} = R_A x - \frac{w_0}{3L}\left(x-\frac{L}{2}\right)^3 \to EI\frac{dy_2}{dx} = R_A\frac{x^2}{2} - \frac{w_0}{12L}\left(x-\frac{L}{2}\right)^4 + C_3$

$\to EIy_2 = R_A\frac{x^3}{6} - \frac{w_0}{60L}\left(x-\frac{L}{2}\right)^5 + C_3x + C_4$

We have 5 unknowns $(C_1, C_2, C_3, C_4, R_A)$ so we need 5 equations Boundary/continuity equations:

(1) $y_1(x=0) = 0$

(2) $y_2(x=L) = 0$

(3) $y_1\left(x=\frac{L}{2}\right) = y_2\left(x=\frac{L}{2}\right)$

(4) $\frac{dy_1}{dx}\left(x=\frac{L}{2}\right) = \frac{dy_2}{dx}\left(x=\frac{L}{2}\right)$

(5) $\frac{dy_2}{dx}(x=L) = 0$

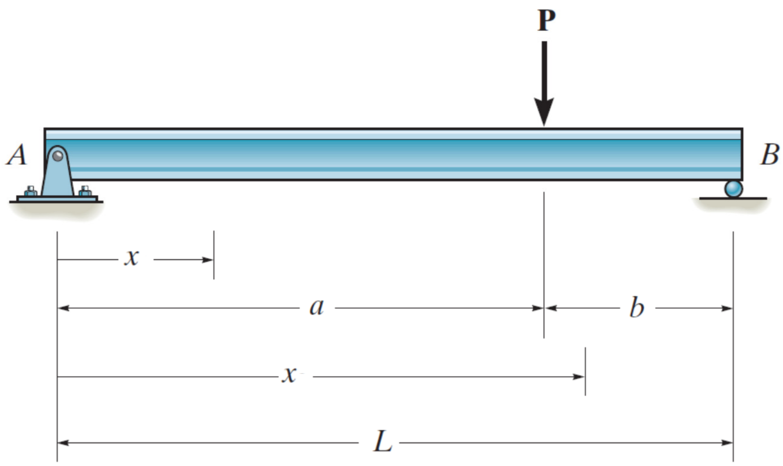
$R_A = \frac{9}{640}w_0L \to From\ equilibrium\ \left(R_A + R_B = \frac{w_0L}{4}\right) we\ can\ find\ R_B = \frac{151}{640}w_0L$

$and\ also\ form\ equilibrium\ (\sum M_B = 0)\ we\ can\ find\ M_B = -\frac{53}{1920}w_0L^2$

$C_1, C_2, C_3, C_4\ are\ also\ obtained\ to\ determine\ the\ elastic\ curve\ of\ deformation.$

Singularity function method

Example 6: Determine the equations of the elastic curve using the singularity function method. EI is constant.



From Statics:

$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

Writing the singularity function to find M(x):

$$w(x) = -R_A\langle x\rangle^{-1} + P\langle x-a\rangle^{-1} \rightarrow V(x) = R_A\langle x\rangle^{0} - P\langle x-a\rangle^{0} \rightarrow M(x) = R_A\langle x\rangle^{1} - P\langle x-a\rangle^{1}$$

$$M(x) = \frac{Pb}{L}\langle x\rangle^{1} - P\langle x-a\rangle^{1}$$

Writing the elastic curve differential equation to find deflection y(x):

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \rightarrow EI\frac{d^2y}{dx^2} = \frac{Pb}{L}\langle x\rangle^{1} - P\langle x-a\rangle^{1} \rightarrow EI\frac{dy}{dx} = \frac{Pb}{2L}\langle x\rangle^{2} - \frac{P}{2}\langle x-a\rangle^{2} + C_1$$

$$\rightarrow EIy = \frac{Pb}{6L}\langle x\rangle^{3} - \frac{P}{6}\langle x-a\rangle^{3} + C_1x + C_2$$

Applying boundary conditions to determine integration constants:

$$(\mathbf{1})\ y(x=0) = 0 \rightarrow\ 0 = 0 - 0 + 0 + C_2 \rightarrow C_2 = 0$$

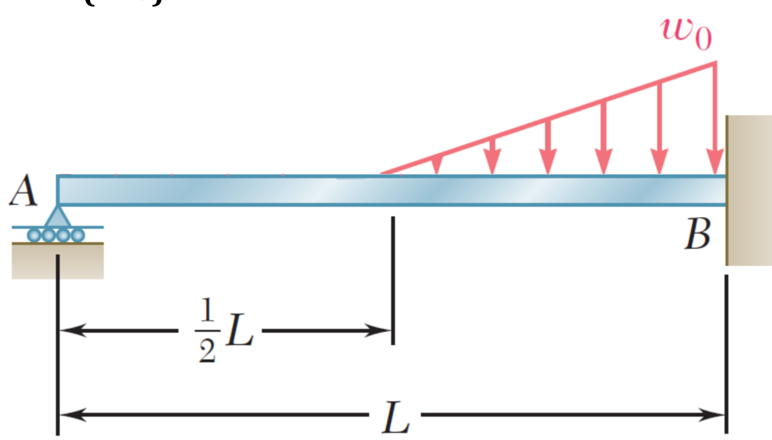
$$(\mathbf{2})\ y(x=L) = 0\ \rightarrow 0 = \frac{Pb(L)^3}{6L} - \frac{P}{6}(L-a)^3 + C_1L \rightarrow C_1 = \frac{Pb^3 - PbL^2}{6L}$$

$$\rightarrow EIy = \frac{Pb}{6L}\langle x\rangle^{3} - \frac{P}{6}\langle x-a\rangle^{3} + \frac{Pb^3 - PbL^2}{6L}x$$

Finding deflection at *x* = *a*:

$$EIy(x=a) = \frac{Pb}{6L}(a)^3 - 0 + \frac{Pb^3 - PbL^2}{6L}a \rightarrow y|_{x=a} = \frac{Pab}{6EIL}(a^2 + b^2 - L^2)$$

Example 7: Determine the equations of the elastic curve using the singularity function method. EI is constant.



Writing the singularity function to find M(x):

$$w(x) = -R_A\langle x\rangle^{-1} + \frac{2w_0}{L}\langle x-\frac{L}{2}\rangle^1 \rightarrow V(x) = R_A\langle x\rangle^0 - \frac{2w_0}{2L}\langle x-\frac{L}{2}\rangle^2$$

$$\rightarrow M(x) = R_A\langle x\rangle^1 - \frac{w_0}{3L}\langle x-\frac{L}{2}\rangle^3$$

Writing the elastic curve differential equation to find deflection y(x):

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \rightarrow EI\frac{d^2y}{dx^2} = R_A\langle x\rangle^1 - \frac{w_0}{3L}\langle x-\frac{L}{2}\rangle^3 \rightarrow EI\frac{dy}{dx} = \frac{R_A}{2}\langle x\rangle^2 - \frac{w_0}{12L}\langle x-\frac{L}{2}\rangle^4 + C_1$$

$$\rightarrow EIy = \frac{R_A}{6}\langle x\rangle^3 - \frac{w_0}{60L}\langle x-\frac{L}{2}\rangle^5 + C_1x + C_2$$

Applying boundary conditions to determine integration constants and R_A:

(1) $y(x=0) = 0 \rightarrow 0 = 0 - 0 + 0 + C_2 \rightarrow C_2 = 0$

(2) $y(x=L) = 0 \rightarrow 0 = \frac{R_A}{6}(L)^3 - \frac{w_0}{60L}(\frac{L}{2})^5 + C_1L$

(3) $\frac{dy}{dx}(x=L) = 0 \rightarrow 0 = \frac{R_A}{2}(L)^2 - \frac{w_0}{12L}(\frac{L}{2})^4 + C_1$

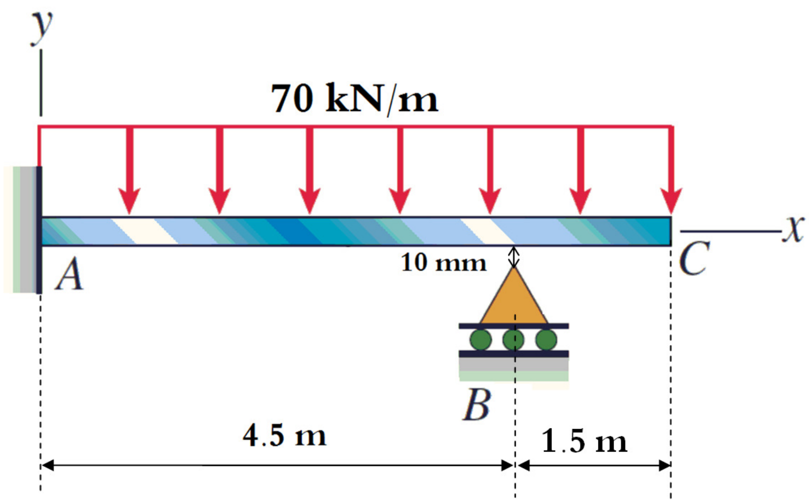
$$\xrightarrow{(2)\,and\,(3)} C_1 = -\frac{7}{3840}w_0L^3 \quad \textbf{and} \quad R_A = \frac{9}{640}w_0L$$

$$\rightarrow EIy = \frac{9}{3840}w_0L\langle x\rangle^3 - \frac{w_0}{60L}\langle x-\frac{L}{2}\rangle^5 - \frac{7}{3840}w_0L^3x$$

Finding deflection at $x = L/2$:

$$EIy = \frac{9}{3840}w_0L(\frac{L}{2})^3 - 0 - \frac{7}{3840}w_0L^3\left(\frac{L}{2}\right) \rightarrow y|_{x=\frac{L}{2}} = -\frac{19}{30720}\frac{w_0L^4}{EI}$$

TBR 2: Beam *ABC* (EI = 70200 kN.m²) is loaded and supported as shown. Prior to the application of the uniform load, there is a gap of 10 mm between the beam and the support *B*. Find deflection at point *C* and reactions at *A* and *B* (1390).



Writing singularity function from right (assuming that support B does not exist):

$$w(x) = 70\langle x\rangle^0 \rightarrow V(x) = -70\langle x\rangle^1 \rightarrow M(x) = -35\langle x\rangle^2 \rightarrow \frac{d^2y}{dx^2} = \frac{M(x)}{EI} \rightarrow EI\frac{d^2y}{dx^2} = -35\langle x\rangle^2$$

$$\rightarrow EI\frac{dy}{dx} = -\frac{35}{3}\langle x\rangle^3 + C_1 \rightarrow EIy = -\frac{35}{12}\langle x\rangle^4 + C_1x + C_2, \qquad y(x=6) = \frac{dy}{dx}(x=6) = 0$$

$$C_1 = 2520, C_2 = -11340 \rightarrow y = \frac{1}{EI}\left\{-\frac{35}{12}\langle x\rangle^4 + 2520x - 11340\right\}$$

$$\rightarrow y(x=1.5) = \frac{1}{70200}\left\{-\frac{35}{12}\langle 1.5\rangle^4 + 2520\times 1.5 - 11340\right\} = -0.108\, m = -108\, mm$$

Therefore the beam AC comes into contact with the support B.

Writing singularity function from right:

$$w(x) = 70\langle x\rangle^0 - R_B\langle x-1.5\rangle^{-1} \rightarrow V(x) = -70\langle x\rangle^1 + R_B\langle x-1.5\rangle^0 \rightarrow M(x) = -35\langle x\rangle^2 + R_B\langle x-1.5\rangle^1$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{M(x)}{EI} \rightarrow EI\frac{d^2y}{dx^2} = -35\langle x\rangle^2 + R_B\langle x-1.5\rangle^1 \rightarrow EI\frac{dy}{dx} = -\frac{35}{3}\langle x\rangle^3 + \frac{R_B}{2}\langle x-1.5\rangle^2 + C_1$$

$$EIy = -\frac{35}{12}\langle x\rangle^4 + \frac{R_B}{6}\langle x-1.5\rangle^3 + C_1x + C_2$$

We have three unknowns ($R_B, C_1, and\ C_2$) so we need tree boundary conditions:

$$y(x=6) = 0, \frac{dy}{dx}(x=6) = 0, y(x=1.5) = -10 = -0.01\, m$$

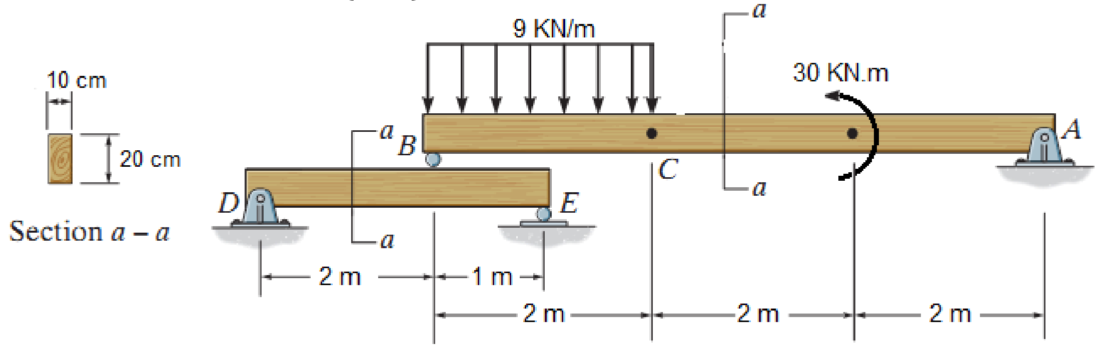
$$R_B = 226.2\, kN, C_1 = 229.1, C_2 = -1031 \rightarrow From\ statics: R_A = 193.8\, kN, M_A = 242.55\, kN$$

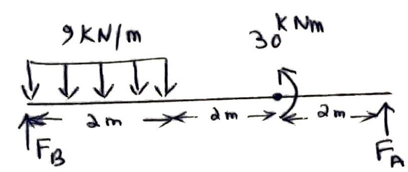
$$y_C = \frac{1}{EI}\left\{-\frac{35}{12}\langle 0\rangle^4 + \frac{226.2}{6}\langle 0-1.5\rangle^3 + 229.1\times 0 - 1031\right\} = -0.0147\, m = -14.7\, mm$$

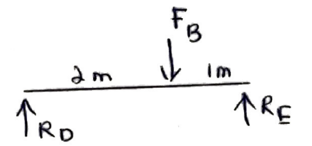
We could write the singularity from right but it would be more difficult as we have three unknowns:

$$w(x) = -R_A\langle x\rangle^{-1} + M_A\langle x\rangle^{-2} + 70\langle x\rangle^0 - R_B\langle x-4.5\rangle^{-1}$$

TBR 3: Find deflection of the wooden beam (E = 10 MPa) at 1 m distance from support A (1391).







$\Sigma M_A = 0 \rightarrow F_B = 20\,kN$

$\Sigma M_E = 0 \rightarrow R_D = \frac{20}{3}$

For beam DE: $W(x) = -R_D\langle x\rangle^{-1} + 20\langle x-2\rangle^{-1}$

$V(x) = R_D\langle x\rangle^{0} - 20\langle x-2\rangle^{0}$

$M(x) = R_D\langle x\rangle^{1} - 20\langle x-2\rangle^{1} = EI\frac{d^2y}{dx^2}$

$\rightarrow EI\,dy/dx = \frac{R_D}{2}\langle x\rangle^2 - \frac{20}{2}\langle x-2\rangle^2 + C_1 \Rightarrow EIy = \frac{R_D}{6}\langle x\rangle^3 - \frac{10}{3}\langle x-2\rangle^3 + C_1x + C_2$

$x=0 \rightarrow y=0$ and $x=3 \rightarrow y=0 \rightarrow C_1 = -\frac{80}{9}$, $C_2 = 0$.

at $x = 2 \rightarrow EIy_B = -\frac{80}{9} \rightarrow y_B = -\frac{80}{9EI}$

For beam AB:

$W(x) = -20\langle x\rangle^{-1} + 9\langle x\rangle^{0} - 9\langle x-2\rangle^{0} + 30\langle x-4\rangle^{-2}$

$V(x) = 20\langle x\rangle^{0} - 9\langle x\rangle^{1} + 9\langle x-2\rangle^{1} - 30\langle x-4\rangle^{-1}$

$M(x) = 20\langle x\rangle^{1} - \frac{9}{2}\langle x\rangle^{2} + \frac{9}{2}\langle x-2\rangle^{2} - 30\langle x-4\rangle^{0} = EI\frac{d^2y}{dx^2}$

$EI\frac{dy}{dx} = 10\langle x\rangle^{2} - \frac{9}{6}\langle x\rangle^{3} + \frac{9}{6}\langle x-2\rangle^{3} - 30\langle x-4\rangle^{1} + C_1$

$EIy = \frac{10}{3}\langle x\rangle^{3} - \frac{3}{8}\langle x\rangle^{4} + \frac{3}{8}\langle x-2\rangle^{4} - 15\langle x-4\rangle^{2} + C_1x + C_2$

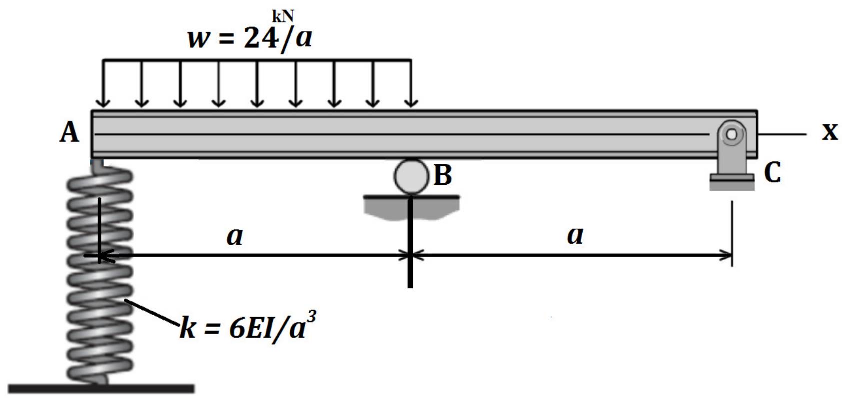
$x=0 \rightarrow y = -\frac{80}{9EI}$, $x=6 \rightarrow y=0$ $\Rightarrow C_1 = \frac{-1175}{27}$, $C_2 = \frac{-80}{9}$

$\therefore EIy_{(x=5)} = \frac{10}{3}(5)^3 - \frac{3}{8}(5)^4 + \frac{3}{8}(5-2)^4 - 15(5-4)^2 - \frac{1175}{27}(5) - \frac{80}{9} \rightarrow EIy_{(x=5)} = -28.82$

$\rightarrow y_{x=5} = \frac{-28.82\times10^3}{EI} = -0.0432\,m = -43.21\,mm = 43.21\downarrow$

$E = 10\times10^9$, $I = \frac{1}{12}(0.1)(0.2)^3$

TBR 4: Find reaction at A as well as deflection of the midpoint of AB (1392).



$$w(x) = -R_A < x >^{-1} + w < x >^0 - w < x-a >^0 - R_B < x-a >^{-1}$$

$$v(x) = R_A < x >^0 - w < x >^1 + w < x-a >^1 + R_B < x-a >^0$$

$$M(x) = R_A < x >^1 - \frac{w}{2} < x >^2 + \frac{w}{2} < x-a >^2 + R_B < x-a >^1 = EI\frac{d^2y}{dx^2}$$

$$EI\frac{dy}{dx} = \frac{R_A}{2} < x >^2 - \frac{w}{6} < x >^3 + \frac{w}{6} < x-a >^3 + \frac{R_B}{2} < x-a >^2 + C_1$$

$$EIy = \frac{R_A}{6} < x >^3 - \frac{w}{24} < x >^4 + \frac{w}{24} < x-a >^4 + \frac{R_B}{6} < x-a >^3 + C_1x + C_2$$

$$EIy = \frac{R_A}{6} < x >^3 - \frac{1}{a} < x >^4 + \frac{1}{a} < x-a >^4 + \frac{R_B}{6} < x-a >^3 + C_1x + C_2$$

$$y_{x=0} = -\frac{R_A}{k} \rightarrow -EI\frac{R_A}{\frac{6EI}{a^3}} = C_2 \rightarrow C_2 = \frac{-a^3R_A}{6}$$

$$y_{x=a} = 0 \rightarrow 0 = \frac{a^3R_A}{6} - a^3 + C_1a - \frac{a^3R_A}{6} \rightarrow C_1 = a^2$$

$$y_{x=2a} = 0 \rightarrow 0 = \frac{8a^3R_A}{6} - 16a^3 + a^3 + \frac{a^3R_B}{6} + 2a^3 - \frac{a^3R_A}{6}$$

$$\rightarrow 7R_A + R_B = 78 \quad (1)$$

From Statics $\sum M_C = 0 \rightarrow R_A(2a) + R_B(a) - w(a)(1.5a) = 0$

$$\rightarrow 2R_A + R_B = 36 \quad (2)$$

$$(1) \quad and \quad (2) \quad \rightarrow \quad R_A = 8.4\ kN$$

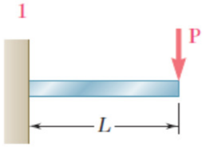
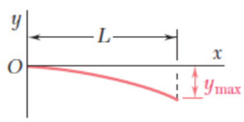
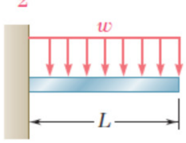
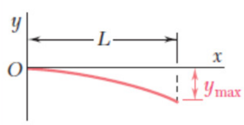
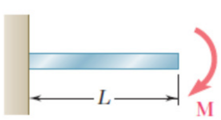
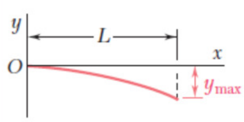
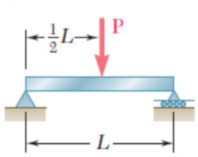
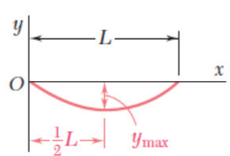
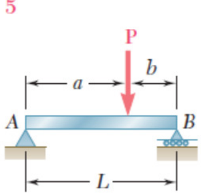
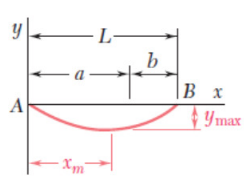
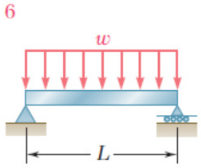
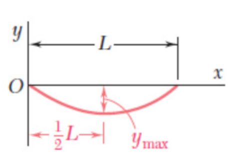
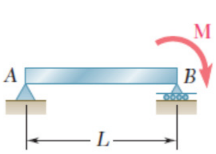
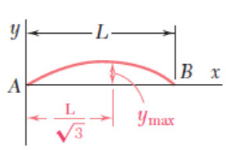
Deflection at mid AB:

$$EIy_{x=a/2} = \frac{R_A}{6} < \frac{a}{2} >^3 - \frac{1}{a} < \frac{a}{2} >^4 + 0 + 0 + C_1\left(\frac{a}{2}\right) + C_2 \rightarrow y_{x=a/2} = \frac{-0.7875\ a^3}{EI}$$

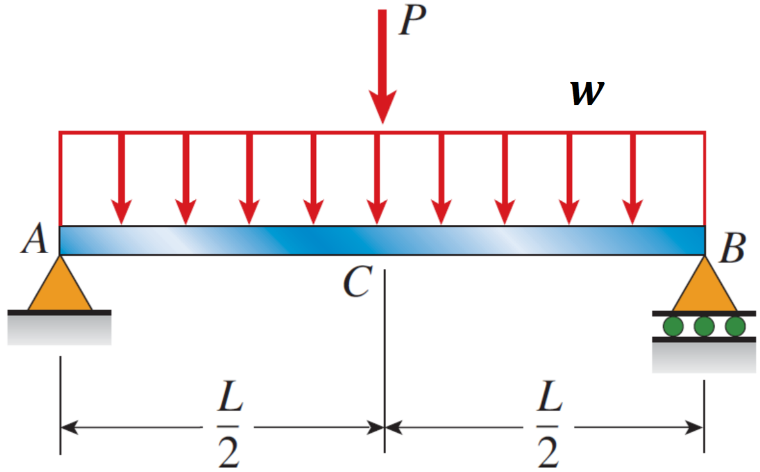
Superposition Method

When a beam is subjected to several concentrated or distributed loads, it is often found convenient to compute separately the slope and deflection caused by each of the given loads. The slope and deflection due to the combined loads are then obtained by applying the principle of superposition and adding the values of the slope or deflection corresponding to the various loads (because the differential equations of the deflection curve are linear). The procedure is facilitated by tables of solutions for common types of loadings and supports.

APPENDIX D Beam Deflections and Slopes

Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
1 		$-\dfrac{PL^3}{3EI}$ **(1)**	$-\dfrac{PL^2}{2EI}$ **(2)**	$y = \dfrac{P}{6EI}(x^3 - 3Lx^2)$
2 		$-\dfrac{wL^4}{8EI}$ **(3)**	$-\dfrac{wL^3}{6EI}$ **(4)**	$y = -\dfrac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
3 		$-\dfrac{ML^2}{2EI}$ **(5)**	$-\dfrac{ML}{EI}$ **(6)**	$y = -\dfrac{M}{2EI}x^2$
4 		$-\dfrac{PL^3}{48EI}$ **(7)**	$\pm\dfrac{PL^2}{16EI}$ **(8)**	For $x \le \frac{1}{2}L$: $y = \dfrac{P}{48EI}(4x^3 - 3L^2x)$
5 		**(9)** For $a > b$: $-\dfrac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EIL}$ at $x_m = \sqrt{\dfrac{L^2 - b^2}{3}}$	**(10)** $\theta_A = -\dfrac{Pb(L^2 - b^2)}{6EIL}$ $\theta_B = +\dfrac{Pa(L^2 - a^2)}{6EIL}$	For $x < a$: $y = \dfrac{Pb}{6EIL}[x^3 - (L^2 - b^2)x]$ For $x = a$: $y = -\dfrac{Pa^2b^2}{3EIL}$
6 		$-\dfrac{5wL^4}{384EI}$ **(11)**	$\pm\dfrac{wL^3}{24EI}$ **(12)**	$y = -\dfrac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$
7 		$\dfrac{ML^2}{9\sqrt{3}EI}$ **(13)**	$\theta_A = +\dfrac{ML}{6EI}$ **(14)** $\theta_B = -\dfrac{ML}{3EI}$	$y = -\dfrac{M}{6EIL}(x^3 - L^2x)$

Example 8: Determine deflection of the beam shown at C as well as slope at point A using the method of superposition.



$$y_C = y_{(1)} + y_{(2)}$$

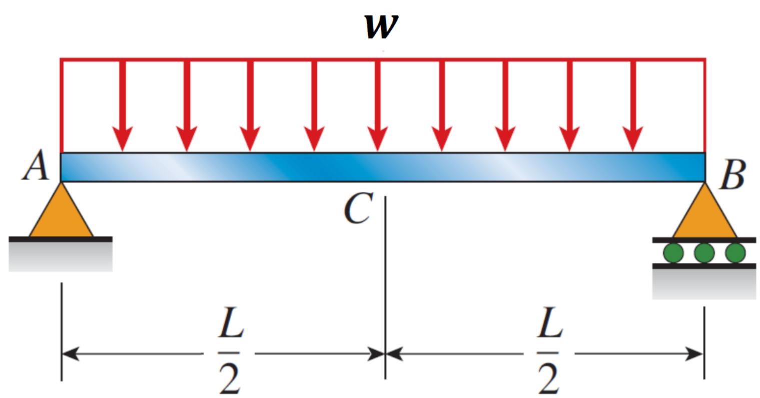
$$y_{(1)} = -\frac{5wL^4}{384EI}$$

$$y_{(2)} = -\frac{PL^3}{48EI}$$

$$y_C = -\frac{5wL^4}{384EI} - \frac{PL^3}{48EI}$$

$$y_C = \frac{5wL^4}{384EI} + \frac{PL^3}{48EI} \downarrow$$

=



(1)

$$\theta_A = \theta_{(1)} + \theta_{(2)}$$

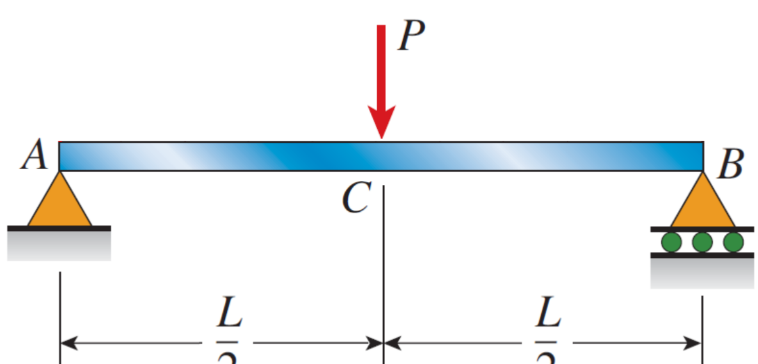
$$\theta_{(1)} = -\frac{wL^3}{24EI}$$

$$\theta_{(2)} = -\frac{PL^2}{16EI}$$

$$\theta_A = -\frac{wL^3}{24EI} - \frac{PL^2}{16EI}$$

$$\theta_A = \frac{wL^3}{24EI} + \frac{PL^2}{16EI}$$

+

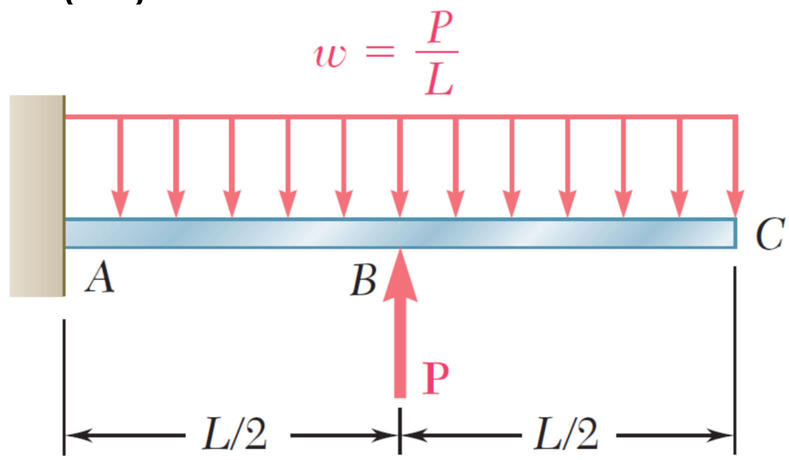


(2)

Using Singularity Function:

$$w(x) = -R_A\langle x\rangle^{-1} + w\langle x\rangle^0 + P\left\langle x - \frac{L}{2}\right\rangle^{-1}$$

Example 9: Determine deflection and slope of the beam shown at C using the method of superposition.



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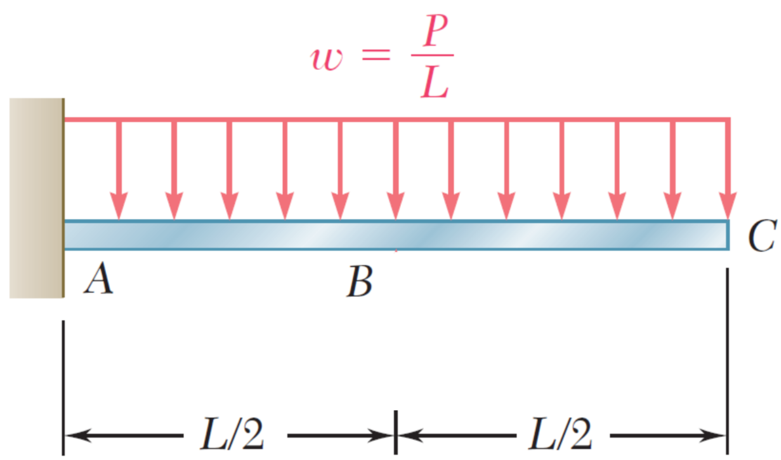
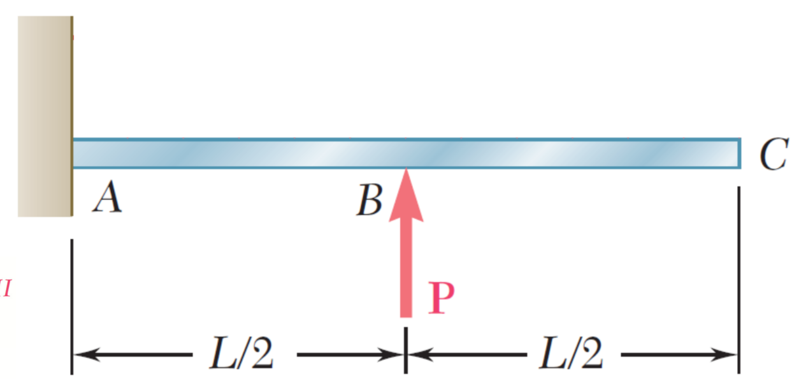


(1)

+



(2)

$$\theta_C = \theta_{(C1)} + \theta_{(C2)}$$

$$\theta_{(C1)} = -\frac{wL^3}{6EI} = -\frac{PL^2}{6EI}$$

$$\theta_{(B2)} = +\frac{P\left(\frac{L}{2}\right)^2}{2EI} = \frac{PL^2}{8EI}$$

For BC in **(2)** we have $M_{BC} = 0$:

$$\frac{d^2y_{BC}}{dx^2} = \frac{M_{BC}}{EI} = 0 \rightarrow \frac{dy_{BC}}{dx} = \theta_{BC} = constant$$

$$\rightarrow \theta_{(C2)} = \theta_{(B2)} = \frac{PL^2}{8EI}$$

$$\theta_C = -\frac{PL^2}{6EI} + \frac{PL^2}{8EI} = -\frac{PL^2}{24EI} \circlearrowleft = \frac{PL^2}{24EI} \circlearrowright$$

$$y_C = y_{(C1)} + y_{(C2)}$$

$$y_{(C1)} = -\frac{wL^4}{8EI} = -\frac{PL^3}{8EI}$$

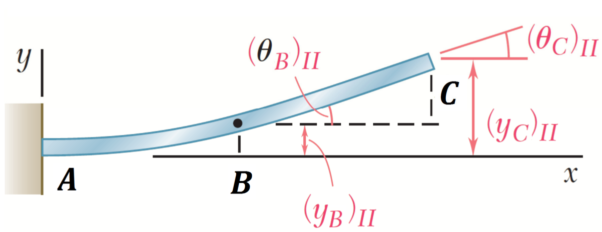
$$y_{(B2)} = +\frac{P\left(\frac{L}{2}\right)^3}{3EI} = \frac{PL^3}{24EI}$$

$$\frac{dy_{BC}}{dx} = \theta_{BC} = constant\ (no\ bending)$$

$$\rightarrow y_{(C2)} - y_{(B2)} = \theta_{BC}\,(x_C - x_B)$$

$$y_{(C2)} = \frac{PL^3}{24EI} + \frac{PL^2}{8EI}\left(\frac{L}{2}\right) = \frac{5PL^3}{48EI}$$

$$y_C = -\frac{PL^3}{8EI} + \frac{5PL^3}{48EI} = -\frac{PL^3}{48EI} \uparrow = \frac{PL^3}{48EI} \downarrow$$



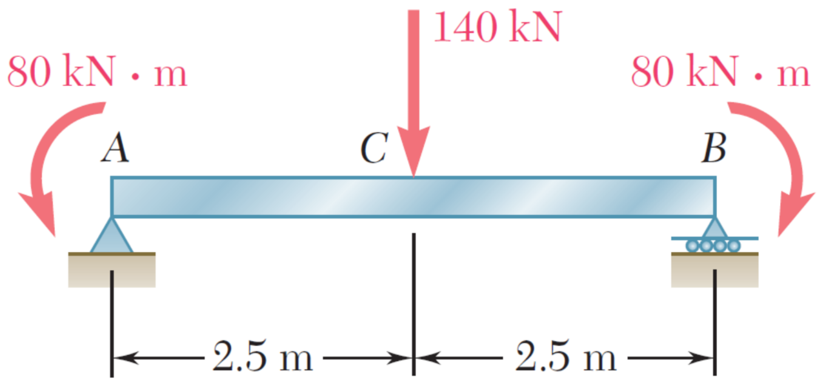
Using Singularity Function:

$$w(x) = -R_A\langle x\rangle^{-1} + M_A\langle x\rangle^{-2} + w\langle x\rangle^0 - P\left\langle x - \frac{L}{2}\right\rangle^{-1} \quad \textbf{\textit{OR}}:$$

$$w(x) = w\langle x\rangle^0 - P\left\langle x - \frac{L}{2}\right\rangle^{-1}$$

(125)

TBR 5: For the beam and loading shown, determine (a) the slope at end A, (b) the deflection at point C. Use the method of superposition.



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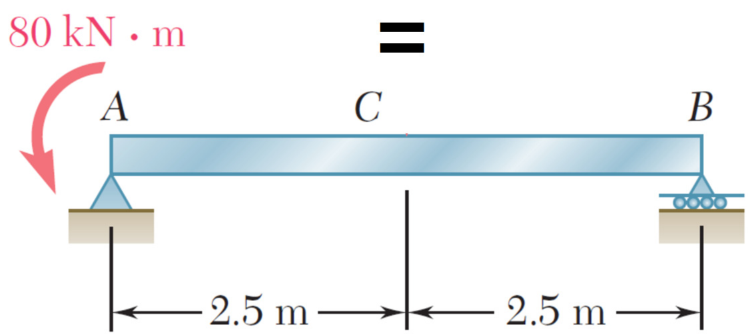
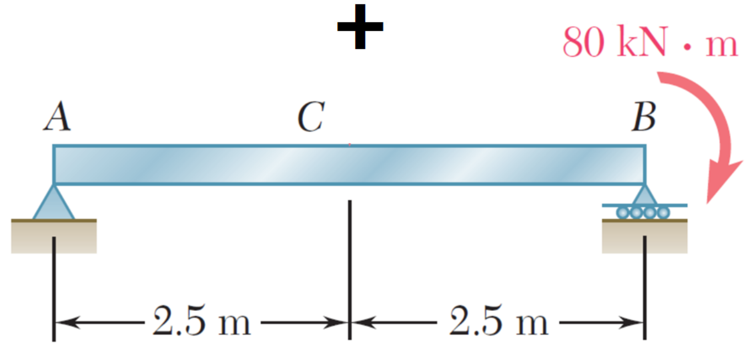


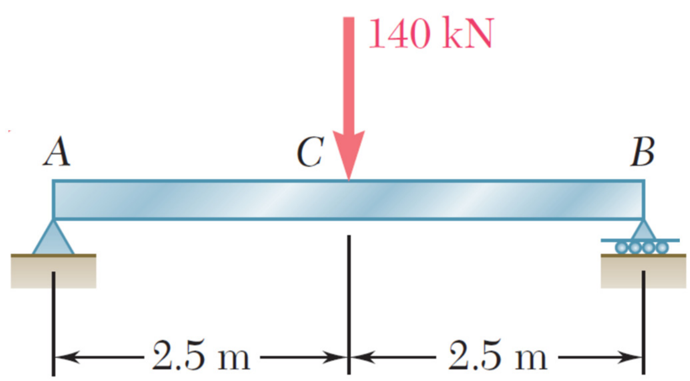
(1)

+



(2)

+



(3)

$$\theta_A = \theta_{(A1)} + \theta_{(A2)} + \theta_{(A3)}$$

$$\theta_{(A1)} = +\frac{ML}{3EI} = \frac{(80\ kNm)(5\ m)}{3EI} = \frac{133.33}{EI}$$

$$\theta_{(A2)} = +\frac{ML}{6EI} = \frac{(80\ kNm)(5\ m)}{6EI} = \frac{66.67}{EI}$$

$$\theta_{(A3)} = -\frac{PL^2}{16EI} = \frac{-(140\ kNm)(5\ m)^2}{16EI} = -\frac{218.75}{EI}$$

$$\theta_A = \frac{133.33}{EI} + \frac{66.67}{EI} - \frac{218.75}{EI} = -\frac{18.75}{EI} \circlearrowleft = \frac{18.75}{EI} \circlearrowright$$

$$y_C = y_{(C1)} + y_{(C2)} + y_{(C3)}$$

$$y_{(C1)} = -\frac{ML}{6EIL}(x^3 - L^2x) = \frac{-80\ kNm}{6EI(5)}(2.5^3 - 5^2 \times 2.5) = +\frac{125}{EI}$$

$$y_{(C2)} = +\frac{125}{EI}$$

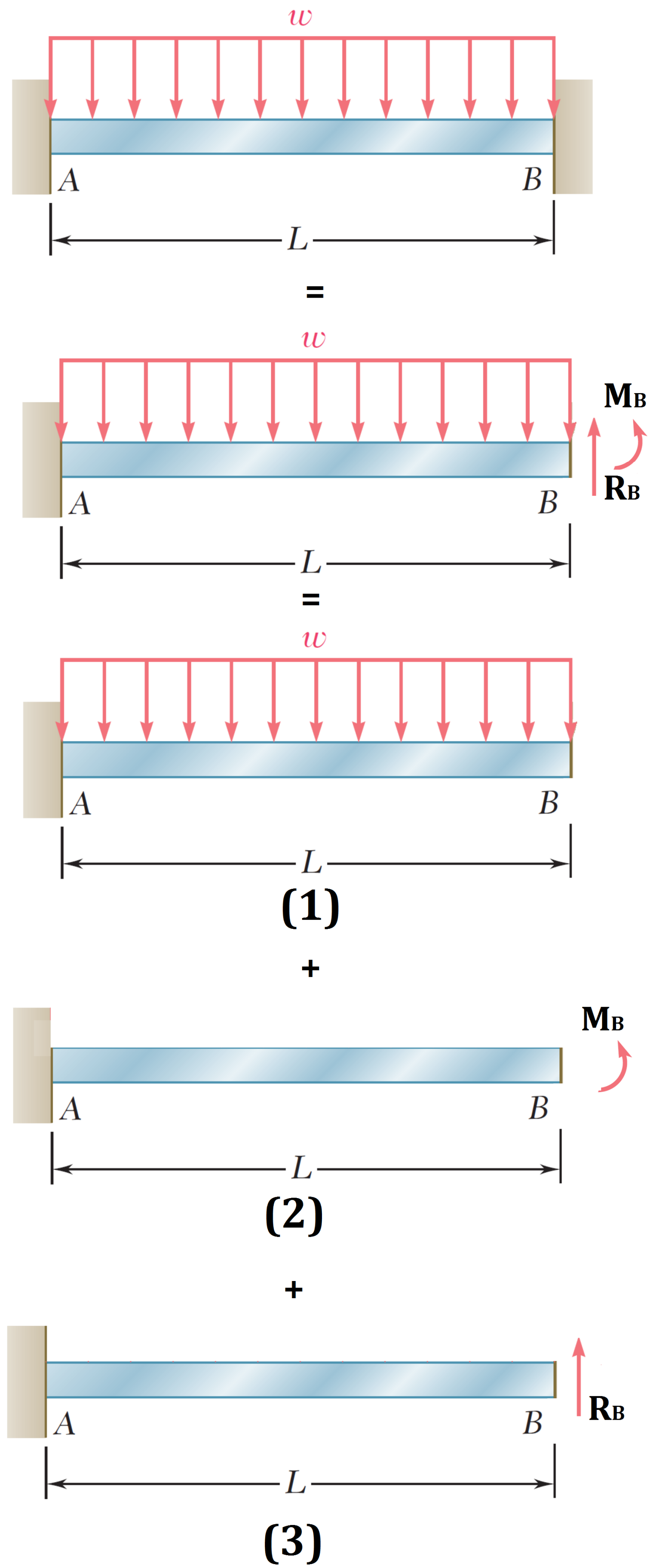
$$y_{(C3)} = -\frac{PL^3}{48EI} = \frac{-(140\ kNm)(5\ m)^3}{48EI} = -\frac{364.58}{EI}$$

$$y_C = \frac{125}{EI} + \frac{125}{EI} - \frac{364.58}{EI} = -\frac{114.58}{EI} \uparrow = \frac{114.58}{EI} \downarrow$$

Using Singularity Function:

$$w(x) = -R_A\langle x\rangle^{-1} + 80\langle x\rangle^{-2} + 140\langle x - 25\rangle^{-1}\ \left(\frac{kN}{m}\right)$$

Example 10: For the beam and loading shown, find reaction at A using the method of superposition.



The system is indeterminate statically (order 2) so two equations (apart from equilibrium equations) are required:

$$\theta_B = 0 \ \ and \ \ y_B = 0$$

$$\theta_B = \theta_{(B1)} + \theta_{(B2)} + \theta_{(B3)}$$

$$\theta_{(B1)} = -\frac{wL^3}{6EI}$$

$$\theta_{(B2)} = \frac{M_B L}{EI}$$

$$\theta_{(B3)} = \frac{R_B L^2}{2EI}$$

$$\theta_B = -\frac{wL^3}{6EI} + \frac{M_B L}{EI} + \frac{R_B L^2}{2EI} = 0 \qquad (1)$$

$$y_B = y_{(B1)} + y_{(B2)} + y_{(B3)}$$

$$y_{(B1)} = -\frac{wL^4}{8EI}$$

$$y_{(B2)} = \frac{M_B L^2}{2EI}$$

$$y_{(B3)} = \frac{R_B L^3}{3EI}$$

$$y_B = -\frac{wL^4}{8EI} + \frac{M_B L^2}{2EI} + \frac{R_B L^3}{3EI} = 0 \qquad (2)$$

$$\xrightarrow{(1)\ and\ (2)} R_B = \frac{wL}{2}, M_B = -\frac{wL^2}{12}$$

Singularity Function:
$w(x) = -R_A\langle x\rangle^{-1} + M_A\langle x\rangle^{-2} + w\langle x\rangle^0$

(127)

Example 11: Determine the deflection at point D of the beam shown using the method of superposition.

$$y_D = y_{(D1)} + y_{(D2)}$$

$$y_{(D1)} = \frac{-5wL^4}{384EI} = \frac{-5w(2a)^4}{384EI}$$
$$= \frac{5wa^4}{24EI} \downarrow$$

$$y_{(D2)} = -\frac{M}{6ELI}(x^3 - L^2x)$$
$$= -\frac{\frac{wa^2}{2}}{6E(2a)I}(a^3 - (2a)^2a)$$
$$= \frac{wa^4}{8EI} \uparrow$$

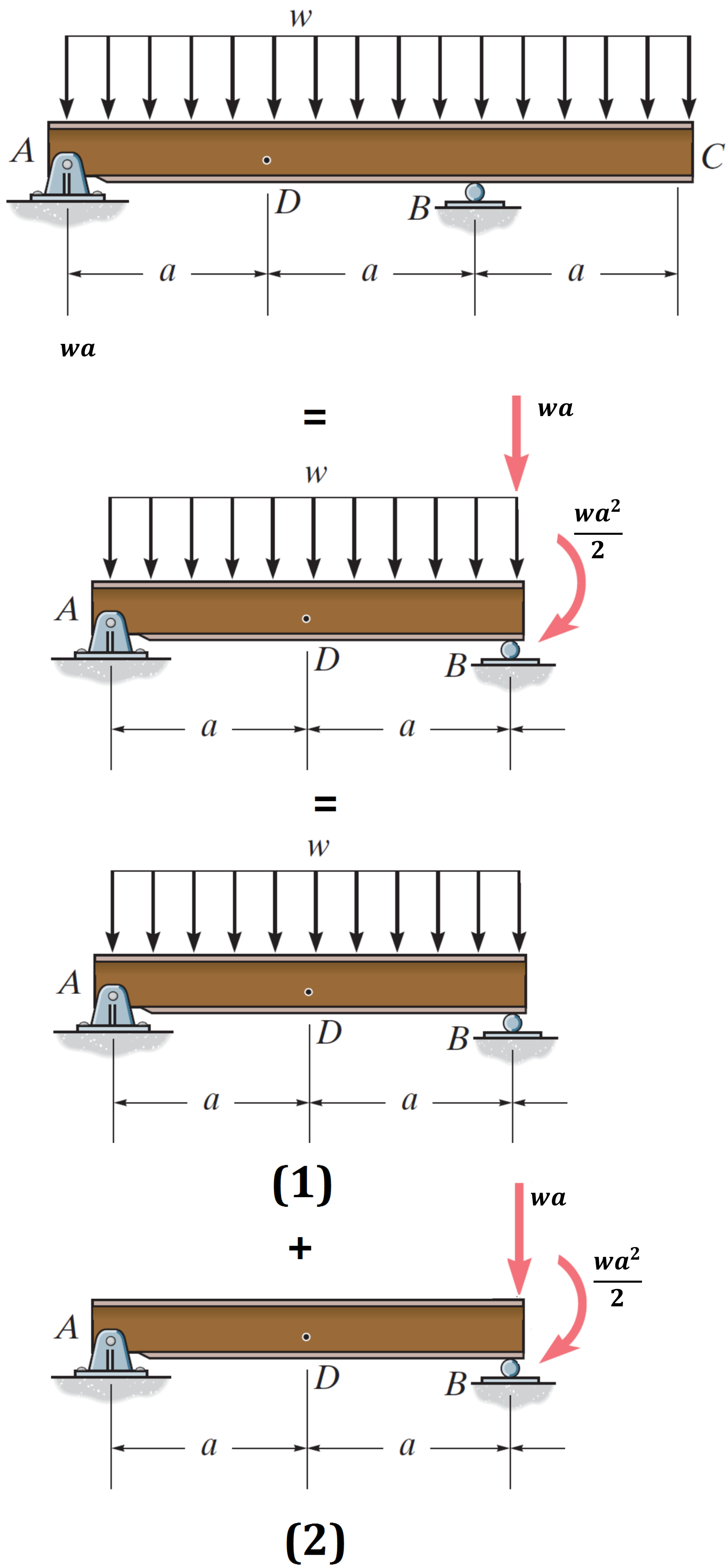
$$y_D = \frac{5wa^4}{24EI} \downarrow + \frac{wa^4}{8EI} \uparrow = \frac{wa^4}{12EI} \downarrow$$

Show that:

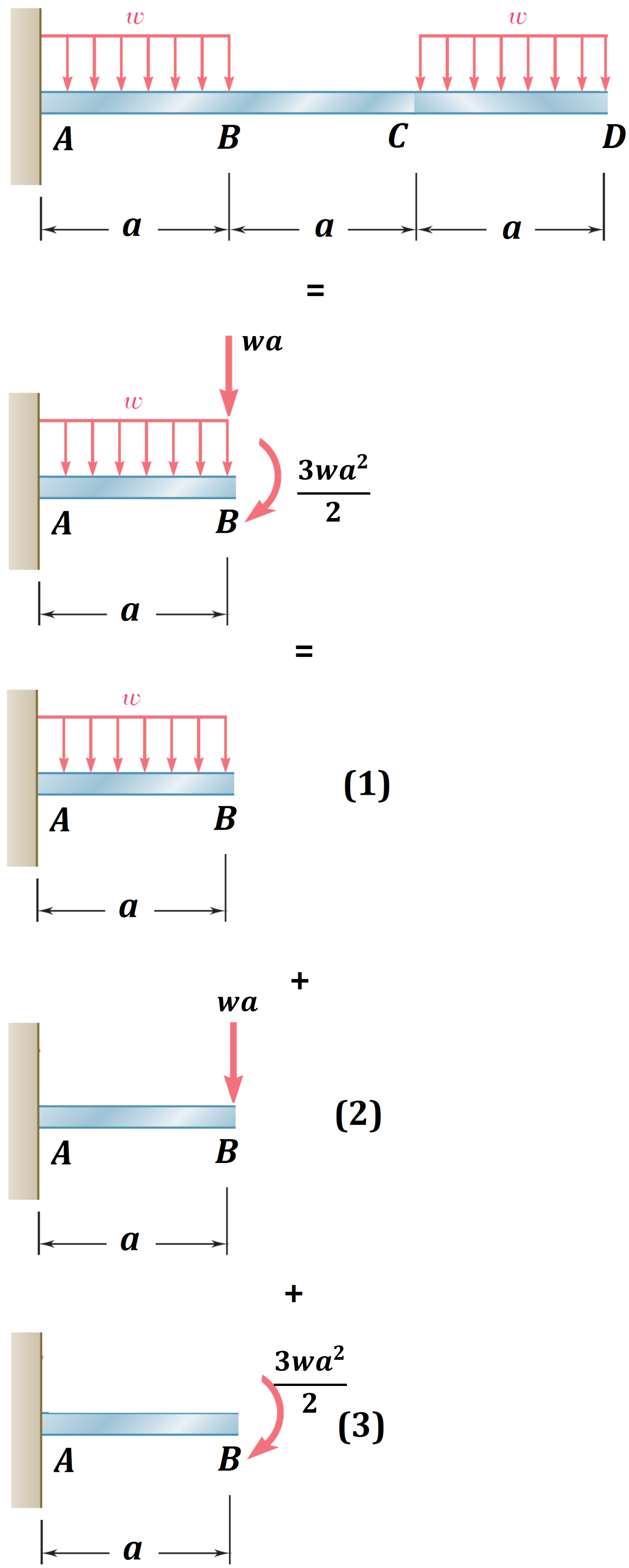
$$\theta_A = \frac{wa^3}{6EI}$$

Singularity Function:

$$w(x) = w\langle x\rangle^0 - R_B\langle x - a\rangle^{-1}$$



Example 12: Determine the deflection and slope at point B of the beam shown using the method of superposition.



$$y_B = y_{(B1)} + y_{(B2)} + y_{(B3)}$$

$$y_B = -\frac{wa^4}{8EI} - \frac{wa\,a^3}{3EI} - \frac{\frac{3wa^2}{2}a^2}{2EI}$$
$$= -\frac{29}{24}wa^4 = \frac{29}{24}wa^4 \downarrow$$

$$\theta_B = \theta_{(B1)} + \theta_{(B2)} + \theta_{(B3)}$$

$$\theta_B = -\frac{wa^3}{6EI} - \frac{wa\,a^2}{2EI} - \frac{\frac{3wa^2}{2}a}{EI}$$
$$= -\frac{13}{6}\frac{wa^3}{EI} \circlearrowleft = \frac{13}{6}\frac{wa^3}{EI} \circlearrowright$$

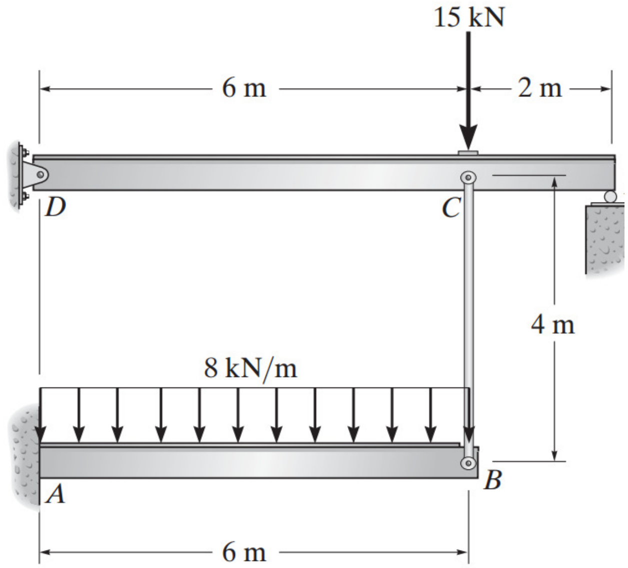
Find deflection at C using superposition method.

Singularity Function:

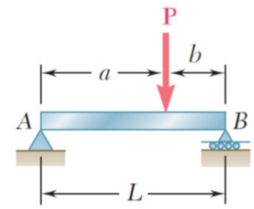
$$w(x) = w\langle x\rangle^0 - w\langle x-a\rangle^0 - w\langle x-2a\rangle^0$$

TBR 6: Find deflection of C using superposition method (1393).

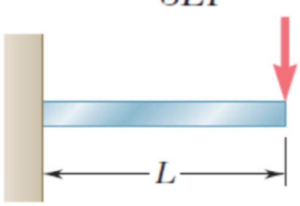
I=100×10⁶ mm⁴ E = 200 GPa A_{BC} = 200 mm²



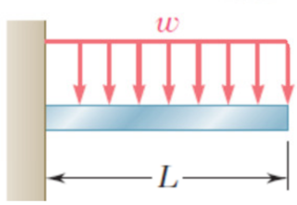
For $x = a$: $y = -\dfrac{Pa^2b^2}{3EIL}$

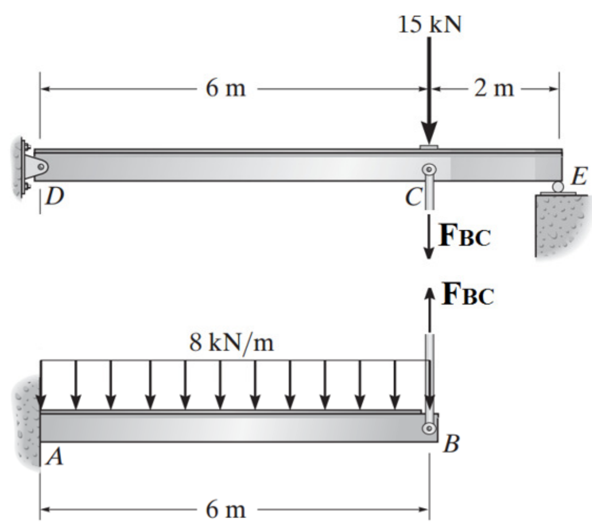


$y_{max} = -\dfrac{PL^3}{3EI}$ 10



$y_{max} = -\dfrac{wL^4}{8EI}$





Statically indeterminate: $\boldsymbol{y_C = y_B + \dfrac{F_{BC}L_{BC}}{A_{BC}E_{BC}}}$ 10

$$\rightarrow \frac{-(15\,kN + F_{BC}) \times (6\,m)^2 \times (2\,m)^2}{3 \times E \times (100 \times 10^{-6}\,m^4) \times 8} = \left(\frac{-8 \times (6\,m)^4}{8 \times E \times (100 \times 10^{-6}\,m^4)} + \frac{F_{BC} \times (6\,m)^3}{3 \times E \times (100 \times 10^{-6}\,m^4)}\right) + \frac{F_{BC} \times (4\,m)}{200\,mm^2 \times E}$$ 10

$\rightarrow F_{BC} = 15.075\,kN$ 5

$$\rightarrow \delta_C = \frac{-(15000 + 15075)N \times (6\,m)^2 \times (2\,m)^2}{3 \times 200 \times 10^9 \times (100 \times 10^{-6}\,m^4) \times 8} = -9.02\,mm \cong 9\,mm \downarrow$$ 5